# Barcodes for Hamiltonian homeomorphisms of surfaces

Joly Benoît

Ruhr University Bochum

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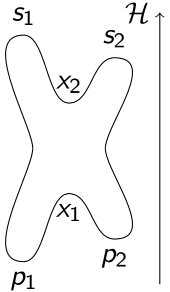


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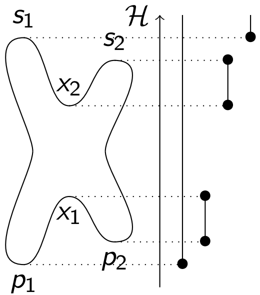


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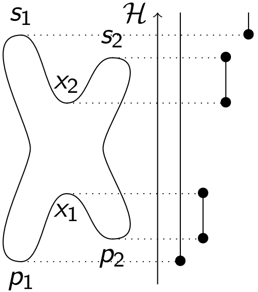


-birth points
$(p_1, p_2, x_2, s_1)$

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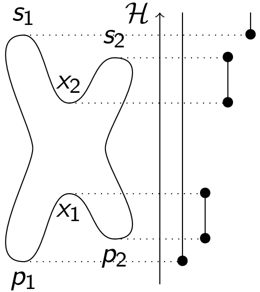


-birth points $(p_1, p_2, x_2, s_1)$

-death points $(x_1, s_2)$

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- Endpoints of bars are the values of the spectrum of $\mathcal{H}$,
- The barcode is a conjugacy invariant,
- The barcodes are $C^0$-continuous and extend to homeomorphisms. (Kislev-Shelukhin, Le Roux-Seyfaddini-Viterbo, Jannaud, Buhovski-Humilière-Seyfaddini)

## Main goal

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## Definition (Hamiltonian homeomorphisms)

An isotopy $I = (f_t)_{t\in[0,1]}$ induces a Hamiltonian homeomorphism if its flux through every closed loop $\gamma \subset \Sigma$ is zero:

$$\int_\Sigma \gamma \wedge I(z)\ \omega = 0,$$

where $I(z) : t \mapsto f_t(z)$

# Context and main difficulties

From now on we consider :

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*Difficulties :*

- There is no function defined everywhere,
- We can not compute directly a filtered homology on $\Sigma$.

# Application $\beta$

## Construction

There exists an application

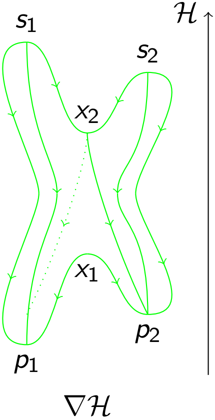
$$\beta : \mathcal{G} \mapsto Barcodes,$$

where $\mathcal{G}$ is the set of couples $(G, A)$ s.t.

- $G$ is a finite oriented and connected graph,
- $A : V \to \mathbb{R}$ decreasing along the edges,

where $V$ is the set of vertices of $G$.

# Morse Barcodes and graphs



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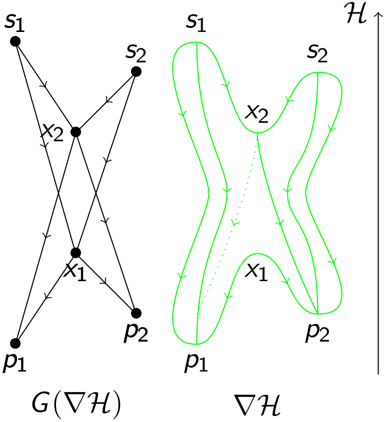
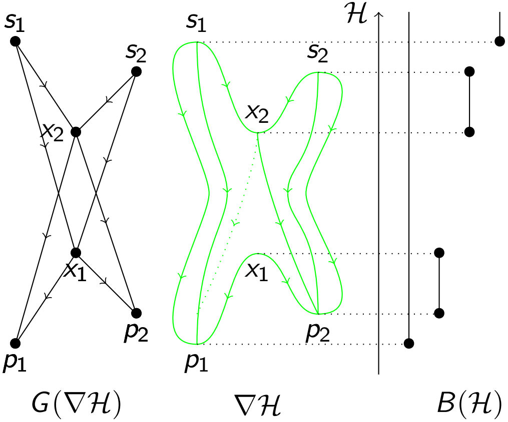



The graph is defined by the gradient lines, $A = \mathcal{H}$.

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# Positively transverse foliations

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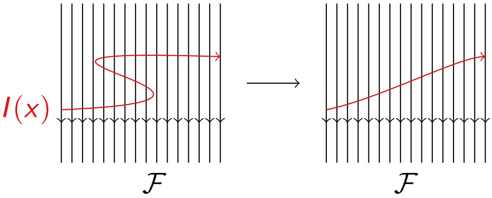
- $I$ is *maximal* if $\mathrm{Sing}(I)$ is maximal for the inclusion,
- An oriented topological foliation $\mathcal{F}$ on $\Sigma\backslash\mathrm{Sing}(I)$ is *positively transverse* if $\forall x \in \Sigma\backslash\mathrm{Sing}(I)$ the path $I(x) : t \mapsto f_t(x)$ is as follows:



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Every foliation $\mathcal{F}$ positively transverse to a maximal isotopy of a Hamiltonian homeomorphism $f$ is *gradient-like*.

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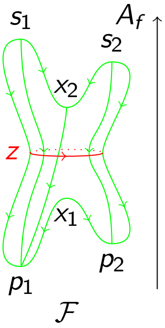


$$\forall \phi \in \mathcal{F},\ A_f(\alpha(\phi)) > A_f(\omega(\phi)).$$

# Construction of a barcode

**Definition**

Let $G(\mathcal{F})$ be a graph where the set of vertices is $\mathrm{Sing}(I)$ and there exists an oriented edge from $x$ to $y$ if there exists a leaf $\phi$ of $\mathcal{F}$ from $x$ to $y$.

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$$(f, I) \rightsquigarrow \mathcal{F} \rightsquigarrow (G(\mathcal{F}), A_f) \overset{\beta}{\mapsto} B(\mathcal{F}) \subset \mathit{Barcodes},$$

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## Theorem

*$B(\mathcal{F})$ is independent of $\mathcal{F}$, it depends only on $I$.*

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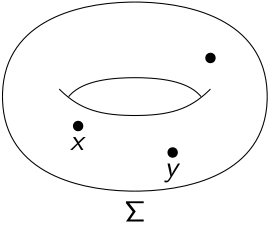

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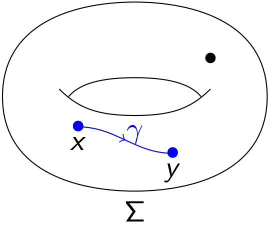

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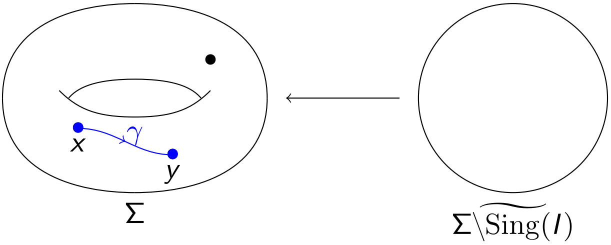

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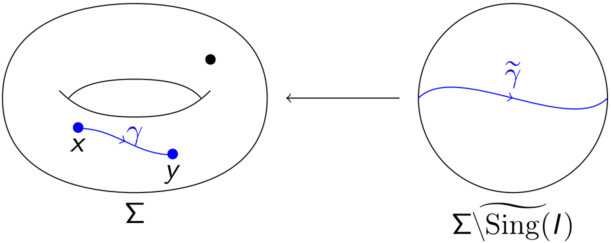

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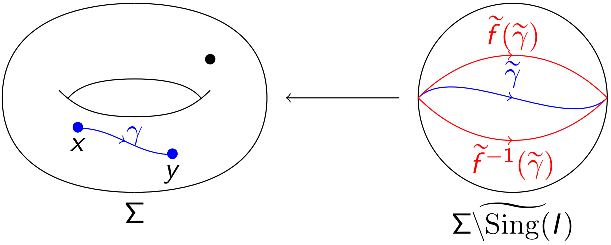

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# Equality of the barcodes

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**Summary**

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### Question 1

Can we construct barcodes which depend only on $f$?

# Do we obtain Floer Homology barcodes?

**Theorem 2 (J.)**

Let $f \in \mathrm{Ham}(\Sigma)$ be $C^2$-close to the identity s.t the fixed points are nondegenerate.
For every foliation $\mathcal{F}$ positively transverse to a maximal isotopy, the barcode $B(\mathcal{F})$ is equal to the Floer homology barcode of $f$.

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**Question 2**

Is Theorem 2 more general?

Thank you!