POLYHEDRAL LIOUVILLE DOMAINS

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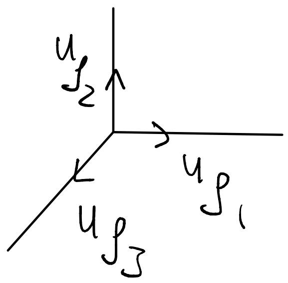
POLYHEDRAL DATA

$d \in \mathbb{N}^+$, $(\mathbb{C}^\times)^d$, coord. $(z_1, \dots, z_d)$

Fan Σ: collection of convex cones $\sigma \subset \mathbb{R}^d$ closed under faces/intersections

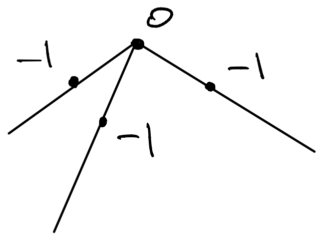
- $\sigma = \mathrm{Conv}(\mathbb{R}_{\geq 0} u_\rho : \rho \in \sigma(1))$, $u_\rho \in \mathbb{Z}^d$
- $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^d$



[proper toric variety $X(\Sigma)$]

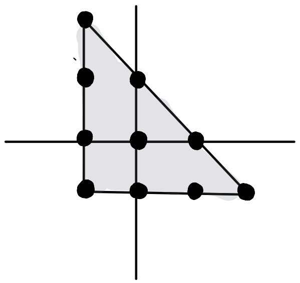
PL function φ: $C^0(\mathbb{R}^d, \mathbb{R})$ with $\varphi|_\sigma$ linear $\forall \sigma \in \Sigma$

- $\sigma \in \Sigma(d) \Rightarrow \varphi|_\sigma = \langle m_\sigma, - \rangle$, $m_\sigma \in \mathbb{Z}^d$
- $\varphi = \min_{\dim(\sigma)=d} \langle m_\sigma, - \rangle$ with no ties



[ample Cartier divisor $D_\varphi = -\sum_{\rho \in \Sigma(1)} \varphi(u_\rho) D_\rho$]

$$(\Sigma, \varphi) \Rightarrow \Delta(\varphi) = \{ m \in \mathbb{R}^d : \langle m, u_\rho \rangle \geq \varphi(u_\rho) \ \forall \rho \in \Sigma(1) \}$$



convex polytope, $m \in \Delta(\varphi) \cap \mathbb{Z}^d$ gives

$\chi^m(z_1, \dots, z_d) = z_1^{m(1)} \cdots z_d^{m(d)}$ character $(\mathbb{C}^\times)^d \to \mathbb{C}^\times$

[basis of $H^0(X(\Sigma), \mathcal{O}(D_\varphi))$]

CONSTRUCTION OF LIOUVILLE DOMAINS

[ADAPTING MCLEAN'S WORK ON COMPLEMENTS OF SNC DIVISORS]

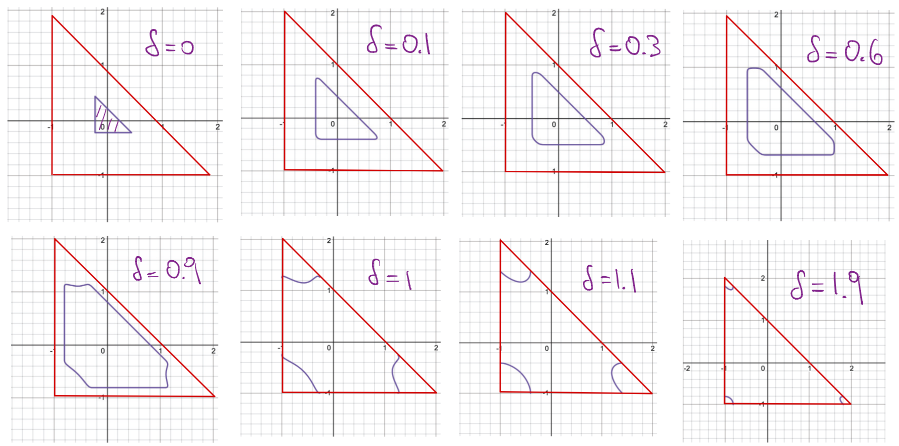
Liouville form: $\theta = \frac{1}{2} d^c F$, $F = \log\Big(\sum_{m \in \Delta(\varphi) \cap \mathbb{Z}^d} |x^m|^2 \Big)$

Hamiltonians: $H_\varepsilon = h_\varepsilon \circ \mu$, $\varepsilon = (\varepsilon_\rho) \in \mathbb{R}^{\Sigma(1)}_{>0}$

$$\mu = \frac{\sum_{m \in \Delta(\varphi) \cap \mathbb{Z}^d} |x^m|^2 m}{\sum_{m \in \Delta(\varphi) \cap \mathbb{Z}^d} |x^m|^2}$$

moment map for $T^d \curvearrowright (\mathbb{C}^\times)^d$

$h_\varepsilon^{-1}(\delta)$

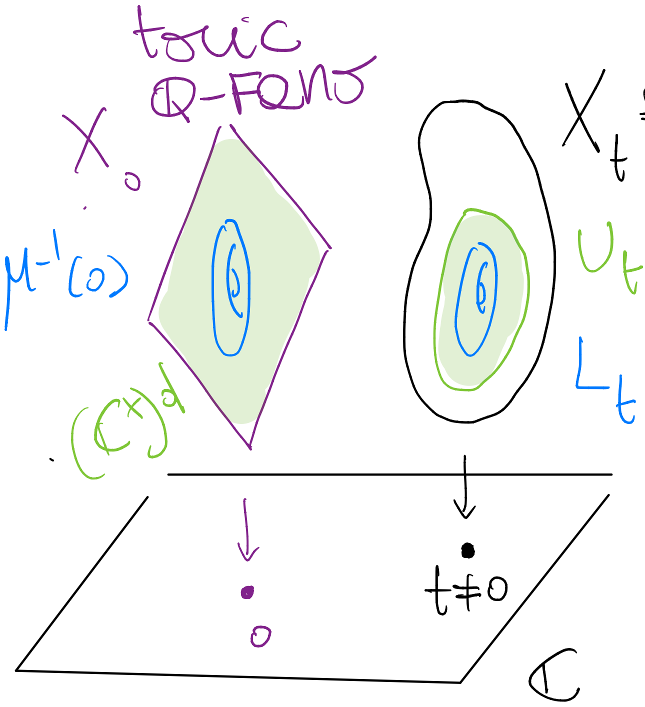


$\partial \Delta(\varphi)$

$\boxed{\text{THM}}$ (C. 2022) Assume $\varphi(u_\rho) < 0$ $\forall \rho \in \Sigma(1)$. Then

1) for $\delta > 0$, $\varepsilon_\rho < -\varphi(u_\rho)$: $H_\varepsilon^{-1}(\delta) \subset (\mathbb{C}^\times)^d$ is of contact type;
2) one-periodic orbits form families B_v^ε, where $v \in \mathbb{Z}^d$ primitive, $v \in \mathring{\sigma}_v$, $B_v \cong$ union of $\mathring{D}^{d-\dim(\sigma_v)} \times T^d$.

MOTIVATION FROM FANO/LG MIRROR SYMMETRY



toric degeneration

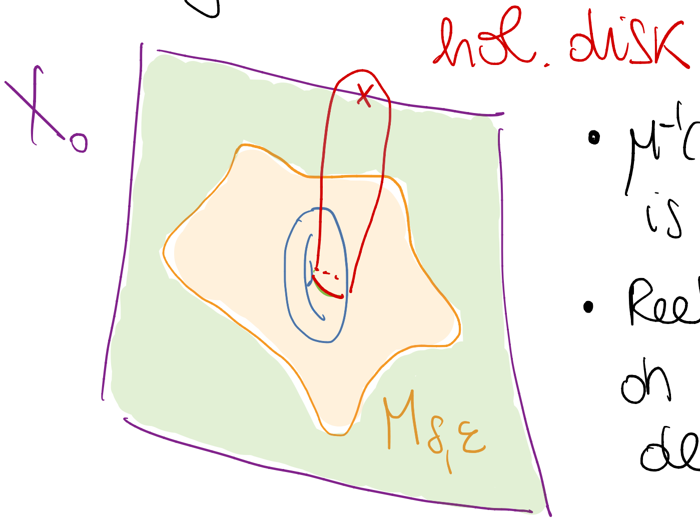
$X_t \cong X$ Fano of $\dim_{\mathbb{C}} X = d$

$$\left\{ \begin{array}{l} \Sigma = \text{fan of } X_0 \\ \forall \rho \in \Sigma(1) : \varphi(u_\rho) = -r \\ \quad \text{with } r >> 0 \end{array} \right.$$

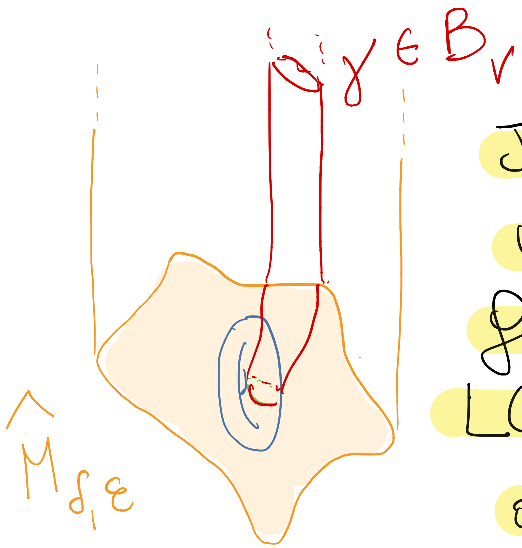
$\swarrow$

Polyhedral data (Σ, φ) gives Liouville subdomains $(\delta < 1)$

$$M_{\delta,\varepsilon} = \{ z \in (\mathbb{C}^\times)^d : H_\varepsilon(z) \leq \delta \}$$



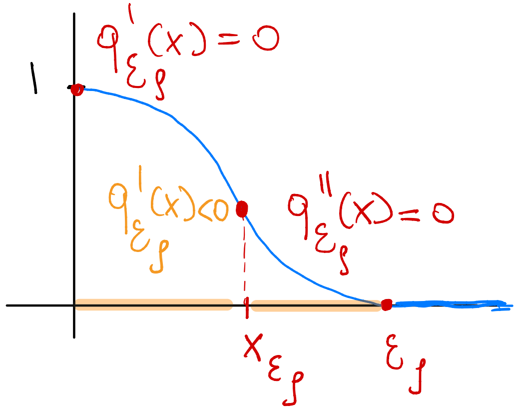
- $\mu^{-1}(0) \subset M_{\delta,\varepsilon}$ is exact
- Reeb dynamics on $\partial M_{\delta,\varepsilon}$ depends on X_0



count of J-hol. cylinders in $\hat{M}_{\delta,\varepsilon}$ should give a local LG potential on mirror $X^\vee$

MORE DETAILS ON THE CONSTRUCTION

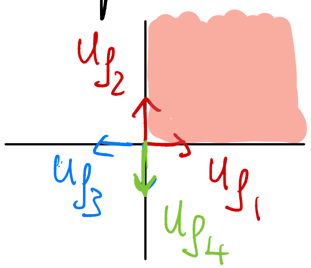
Smoothing function: $h_\varepsilon(m) = \sum_{\rho\in\Sigma(1)} q_{\varepsilon_\rho}(r_\rho(m))$

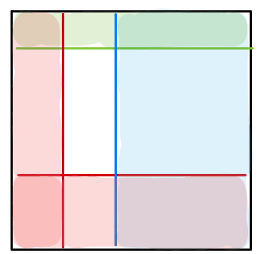
- $r_\rho(m) = \langle m, u_\rho\rangle - \varphi(u_\rho)$ distance from ρ-facet of $\Delta(\varphi)$
- $q_{\varepsilon_\rho}(x) =$ 

bump function with ε_ρ-support

Flow of X_{H_ε}: $X_{H_\varepsilon}(z) = \sum_{\rho\in\sigma(1)} q'_{\varepsilon_\rho}(r_\rho(\mu(z)))\, X_{u_\rho}$ for $z\in S^\varepsilon_\sigma$

- X_{u_ρ} infinitesimal action of one-parameter subgroup $\lambda_{u_\rho}(t) = (t^{u_\rho^{(1)}}, \dots, t^{u_\rho^{(d)}})$
- $\{S^\varepsilon_\sigma\}_{\sigma\in\Sigma}$ partition of $(\mathbb{C}^\times)^d$ in loc. closed sets





Sketch of 1) :

a) $dH_\varepsilon(X_\theta) = \theta(X_{H_\varepsilon}) = \sum_{\rho\in\Sigma(1)} \overbrace{q'_{\varepsilon_\rho}(\tau_\rho\circ\mu)}^{\leq 0}\,\overbrace{\theta(X_{u_\rho})}$

suffices to have $\theta(X_u) < 0$ "near ∞ in ρ-direction" $\forall \rho\in\Sigma(1)$

b) $\theta(X_{u_\rho}) = \langle \mu, u_\rho\rangle + C_\rho$, $C_\rho\in\mathbb{R}$ depending only on $\rho\in\Sigma(1)$

since $(S^1)^d \curvearrowright (\mathbb{C}^\times)^d$ strictly exact Ham action

c) $C_\rho = 0$ $\forall \rho\in\Sigma(1)$, derived from

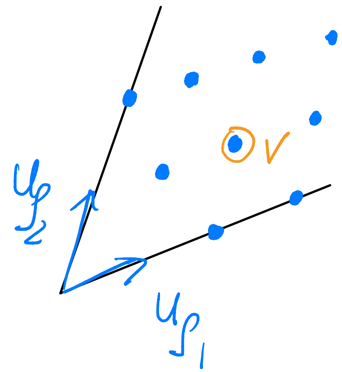
wrapping-averaging formula
$$\int_{H^1=\mathbb{Q}} \lambda^*_{u_\rho}\theta = 2\pi \left\langle \frac{\sum_{m\in\Delta(\varphi)\cap\mathbb{Z}^d} e^{2\langle m, u_\rho\rangle} m}{\sum_{m\in\Delta(\varphi)\cap\mathbb{Z}^d} e^{2\langle m, u_\rho\rangle}}, u_\rho \right\rangle$$

d) $\theta(X_{u_\rho}) = \langle \mu, u_\rho\rangle < \varepsilon_\rho + \varphi(u_\rho) < 0$

"near ∞ in ρ-direction"

for $\varepsilon_\rho < -\varphi(u_\rho)$ >0 by assumption

Sketch of 2): $v \in \mathring{\sigma}_v \cap \mathbb{Z}^d \Rightarrow v = \sum_{\rho \in \sigma_v(1)} d_\rho u_\rho$, $d_\rho \in \mathbb{Q}_{>0}$



a) $B_v^\varepsilon = \{ z \in \mathring{S}_\sigma^\varepsilon : q'_{\varepsilon_\rho}(\mu(z)) = -d_\rho \ \forall \rho \in \sigma_v(1) \}$
manifold of dim $2d - \dim(\sigma_v)$

b) $\gamma \subset B_v^\varepsilon$ has $T(\gamma) = \dfrac{1}{|\mathrm{GCD}(v^{(1)}, \dots, v^{(d)})|}$

TANGENTIAL DIRECTIONS & PICTURES

- Lower bounds on symplectic capacities of polarized projective varieties? (see KAVEH for Gromov width)
- Study of toric singularities using symplectic cohomology? (see EVANS-LEKILI for Du Val)
- $\mathbb{Q}$-Fano ~~toric~~ toric surfaces of Gorenstein index 1 and 2