Singular plane curves and Stable nonsqueezing phenomena

Symplectic Zoominar

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Goal: discuss relationship between

(1) singular rational plane curves

(2) SFT of 4d ellipsoids

(3) symplectic embeddings $E(a,b)\times\mathbb{C}^N \overset{s}{\hookrightarrow} E(a',b')\times\mathbb{C}^N$

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§ Singular algebraic curves

Consider $F(x,y,z) \rightsquigarrow V(F) = \{[x:y:z] \mid F(x,y,z)=0\} \subset \mathbb{CP}^2$

homogeneous polynomial

Recall: $p=[p_0:p_1:p_2]\in V(F)$ is singular

if $\partial_x F(p) = \partial_y F(p) = \partial_z F(p) = 0$.

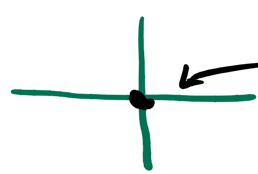
Def: If $0 \in V(F)$, $f(x,y) := F(x,y,1) = f_{(m)}(x,y) + f_{(m+1)}(x,y) + \dots + f_{(d)}(x,y)$,

homogeneous of deg m — homogeneous of deg d

m is the multiplicity of $V(F)$ at 0.

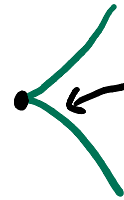
Note: $p\in C \subset \mathbb{CP}^2$ singular $\iff m_p \geq 2$.

Ex: $\underbrace{\{x^2-y^2=0\}}_{(x-y)(x+y),\ \text{so reducible}}$



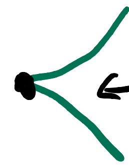
ordinary double pt

Ex: $\{x^3 - y^2 = 0\}$

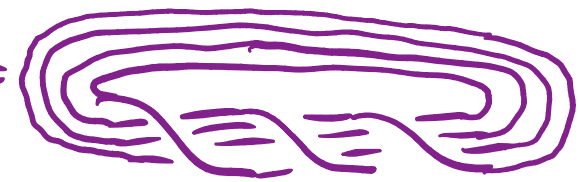


ordinary cusp (multiplicity $= 2$)

Ex: $\{x^p - y^q = 0\}$ $(\gcd(p,q) = 1)$



(p,q) cusp $=$ cone over $T_{p,q}$ torus knot "$\mathfrak{S}_{p,q}$" (multiplicity $= \min(p,q)$)

Ex: $T_{4,3} =$ 

Note: $C := \{X^p - Y^q Z^{p-q} = 0\} \subset \mathbb{CP}^2$ has two singularities:

(1) a $\mathfrak{S}_{p,q}$ at $[0:0:1]$ (2) a $\mathfrak{S}_{p,p-q}$ at $[0:1:0]$

Moreover it is <u>rational</u>:

$$\begin{aligned} \mathbb{CP}^1 &\longrightarrow C \\ [s:t] &\longmapsto [s^q t^{p-q} : s^p : t^p] \end{aligned}$$

Ex: $C := \{y^4 - 2y^2x^3 - 4yx^5 + x^6 - x^7 = 0\}$

Newton–Puiseux: C parametrized near 0 by

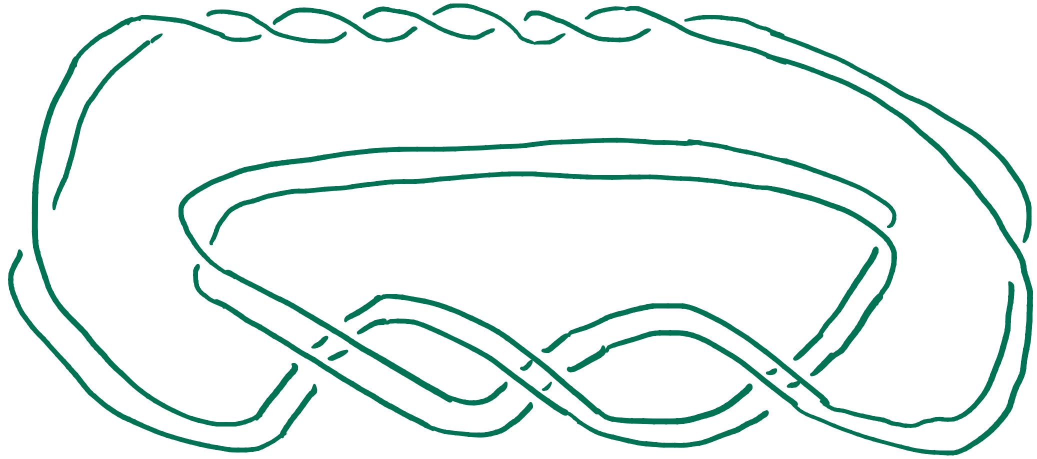
$$t \longmapsto (t^4, t^6 + t^7)$$

$6/4 = 3/2 = p_1/q_1 \rightsquigarrow (p_1, q_1) = (3,2)$

$7/4 = \dfrac{p_2}{q_1 q_2} = \dfrac{p_2}{2 q_2} \rightsquigarrow (p_2, q_2) = (7,2)$

these are the Puiseux pairs of the singularity at 0

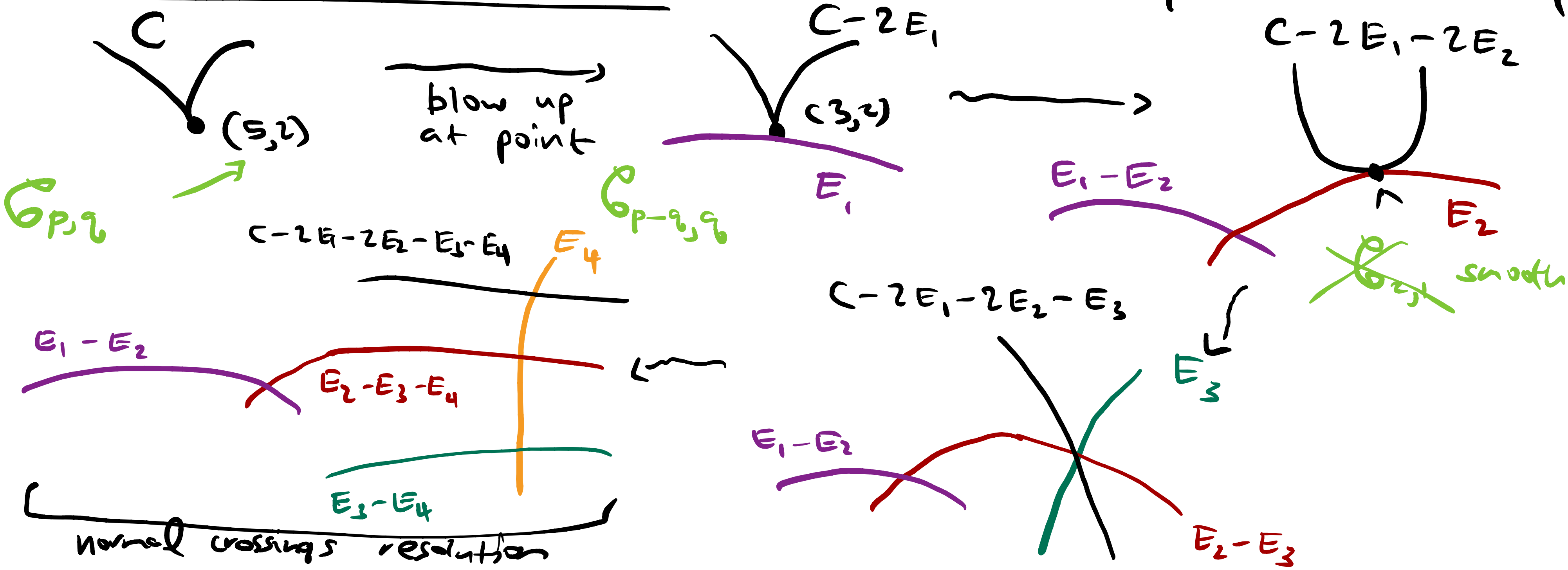
⟶ link
$\parallel$
$C \cap \{|x|^2+|y|^2=\varepsilon\}$
is iterated torus knot:

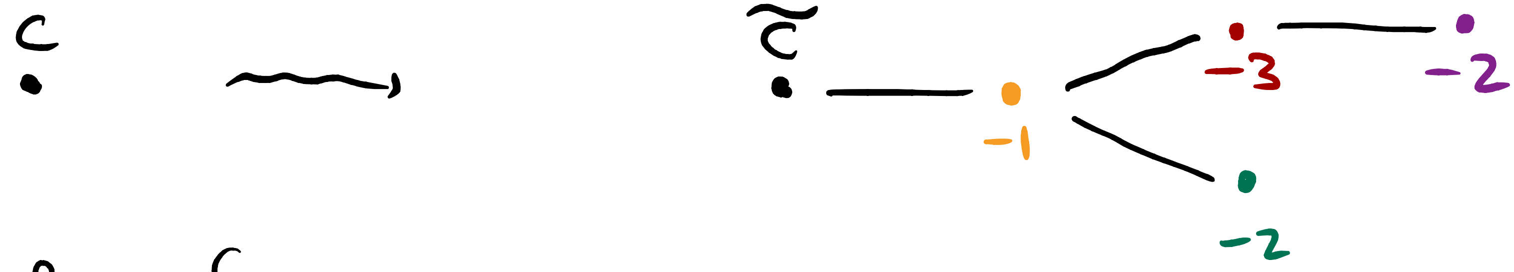


$T(3,2;7,2)$

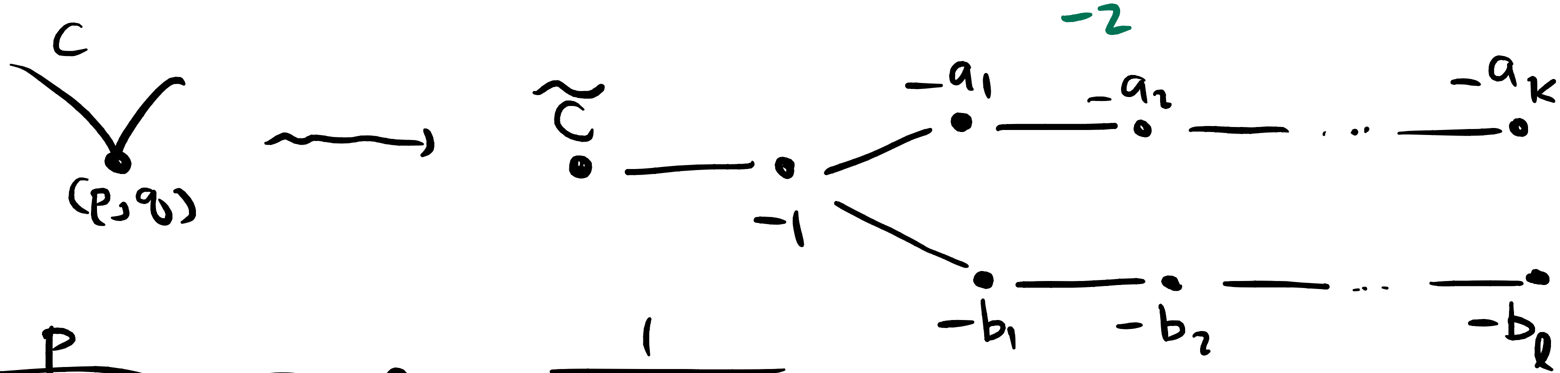
<u>Fact</u>: any unibranched (i.e. locally irreducible) plane curve singularity has link $T(p_1,q_1;\dots;p_k,q_k)$

<u>Resolution of singularities</u>:





In general,



where

$$\frac{p}{p-q^*} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_k}}} \qquad \begin{array}{l} 0 \le q^* < p \\ qq^* \equiv 1 \pmod p \end{array}$$

$$\frac{q}{q-p^*} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cdots - \cfrac{1}{b_\ell}}} \qquad \begin{array}{l} 0 \le p^* < q \\ pp^* \equiv 1 \pmod q \end{array}$$

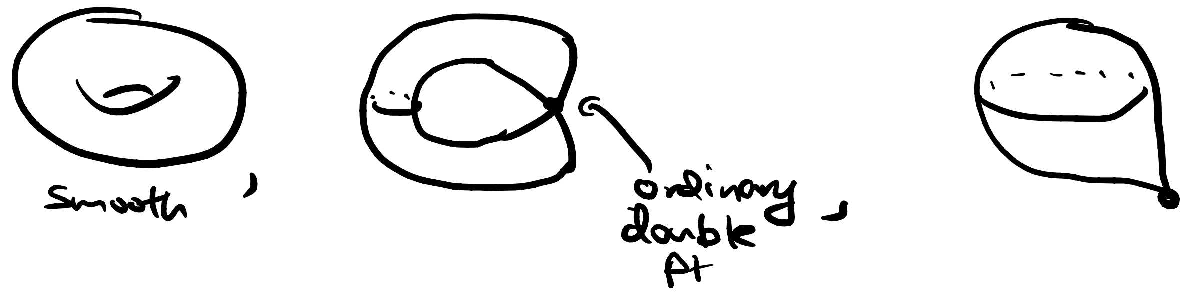
<u>Classification question</u>: What types of singularities can a degree d rational plane curve have?

Adjunction obstruction:

$$\sum_{p \in C \text{ singular pt.}} \delta(p) = \frac{(\deg(C)-1)\cdot(\deg(C)-2)}{2}$$

Here $\delta(\text{ordinary double pt}) = 1$, $\delta(C_{p,q}) = \frac{(p-1)(q-1)}{2}$

Ex: $\deg = 3$



Many other obstructions:
- semigroup obstruction (Heegaard Floer theory),
- semicontinuity of the singularity spectrum (deformation theory)

etc

$\longrightarrow$ Up to degree 7, complete list in the case of only cusp singularity

<u>Theorem</u> (Bodnár et al): $\exists$ a degree d rational plane curve with a $\mathcal{C}_{p,q}$ and no other singularities iff (d, p, q) comes from following list:

(a) $(p,q) = (d-1, d)$

(b) $(p,q) = (d/2, 2d-1)$, d even

(c) $(p,q) = (F_{j-2}^2, F_j^2)$, $d = F_{j-1}^2 + 1 = F_{j-2}F_j$ $j \geq 5$ odd

F_j = jth Fibonacci # ($F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots$)

(d) $(p,q) = (F_{j-2}, F_{j+2})$, $d = F_j$ $j \geq 5$ odd $\longleftarrow$ Orevkov (several constructions)

(e) $(p,q) = (3, 22)$, $d = 8$

(f) $(p,q) = (6, 43)$, $d = 16$

Thm (Orevkov): If C is a degree d rational plane curve with only cusp singularities we have

$$d < \left(\frac{3+\sqrt{5}}{2}\right)\left(\max_{p \in C} m_p + 1\right) + \frac{1}{\sqrt{5}}$$

Thm (Koras – Palka): If C is a rational plane curve with only cusp singularities, then it has ≤ 4 singular pts (and $= 4$ only if $\deg(C) = 5$)

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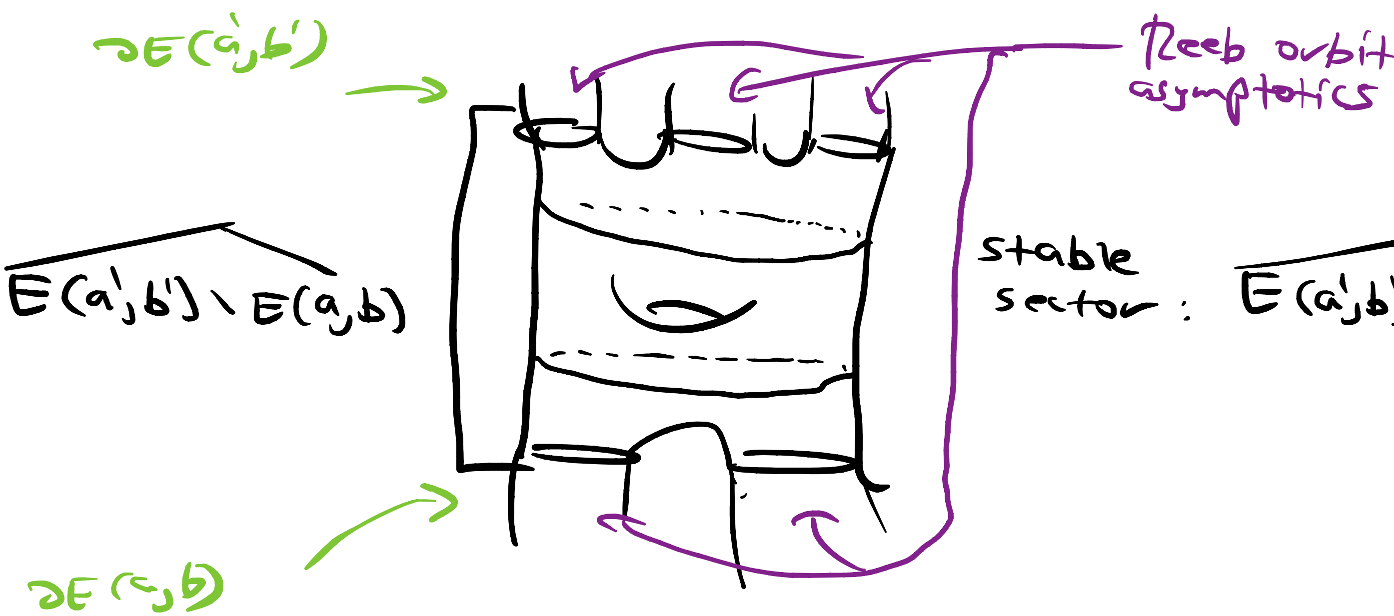
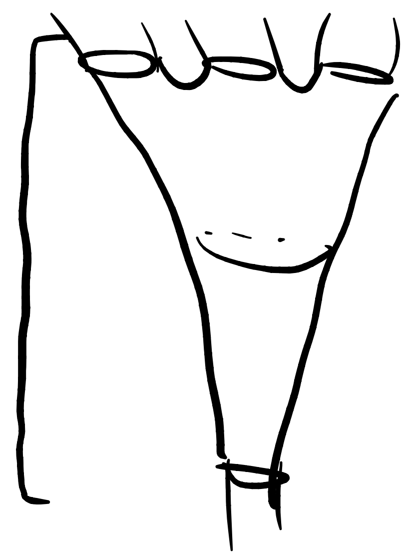
§ SFT of ellipsoids



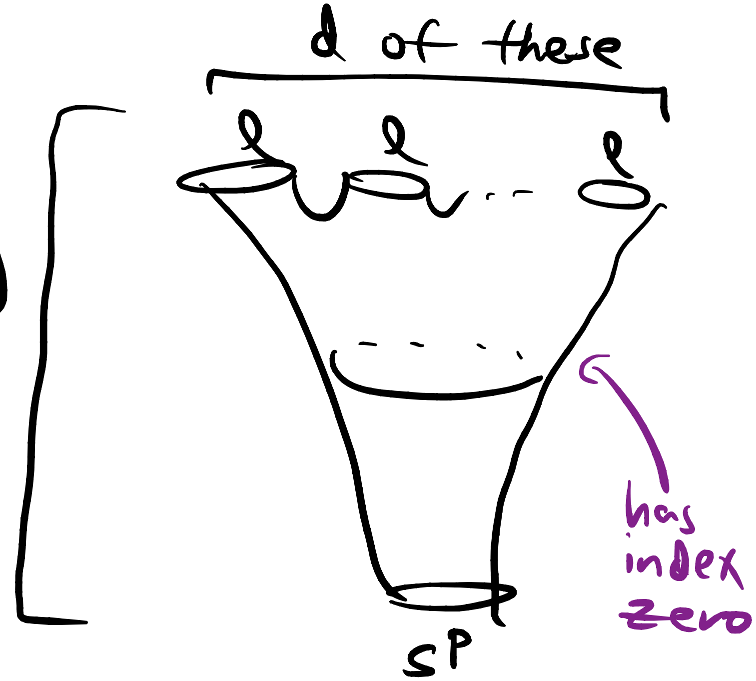
stable sector : $E(a',b') \setminus E(a,b)$

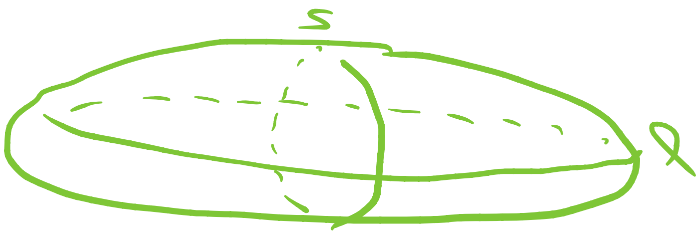


For p, q with $\gcd(p,q)=1$, $d := \frac{p+q}{3} \in \mathbb{Z}$, put

$(p>q)$

$$T_{p,q} = \# \widehat{E(1,1^+) \setminus \varepsilon \cdot E(1, p/q^+)}$$

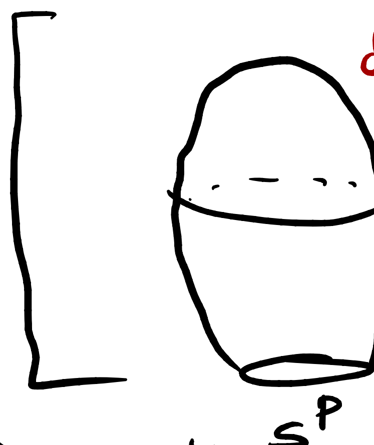




$E(p,q)$

$\wr\wr$

$$\# \widehat{\mathbb{CP}^2 \setminus \varepsilon \cdot E(1, p/q^+)}$$



Can show :

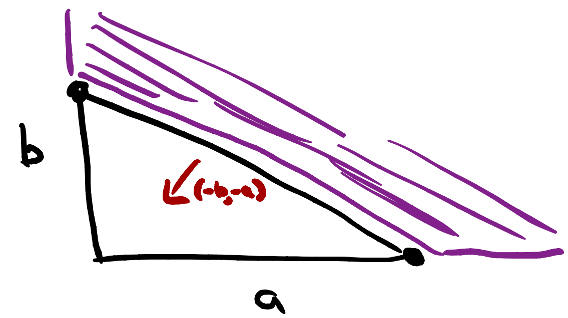
(1) $T_{p,q} \in \mathbb{Z}_{\geq 0}$ is well-defined, independent of all choices

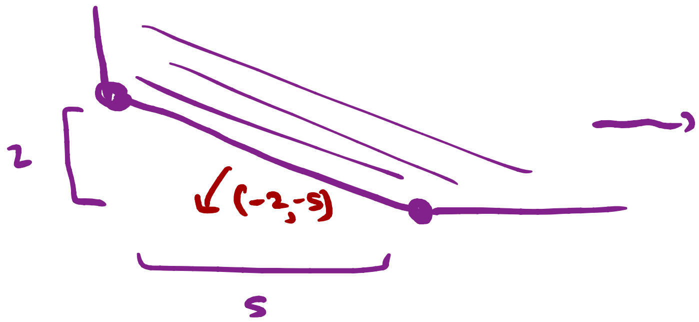
(2) ~~If~~ $T_{p,q} \neq 0$, then Hind's folding embedding $E(1,x) \times \mathbb{C}^N \hookrightarrow B^4\left(\frac{3x}{x+1}\right) \times \mathbb{C}^N$ optimal at $x = p/q$.

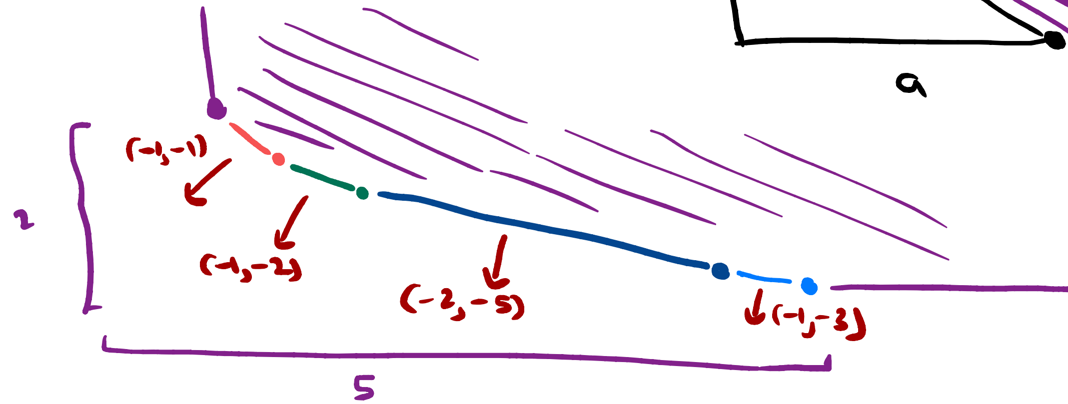
(3) $T_d := T_{3d-1,1}$ is the count of deg d rational plane curves maximally tangent to a generic local divisor

McDuff's ellipsoid resolution:

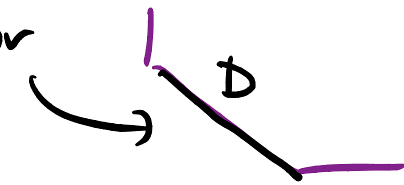
$$E(a,b) = \{\pi|z_1|^2/a + \pi|z_2|^2/b \le 1\}$$





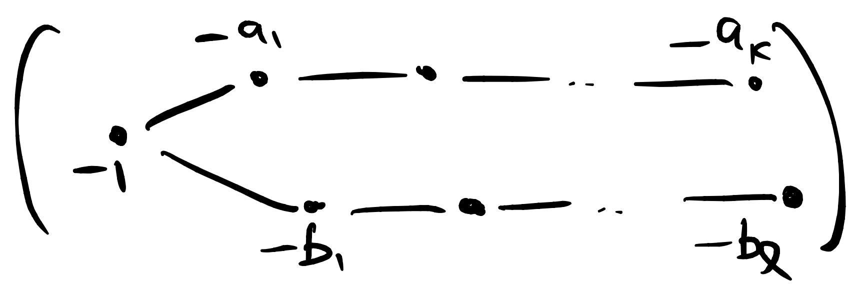


Replaces singular divisor



with chain of smooth divisors

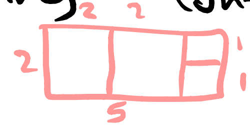
Convex nbd(D) $\approx$ Convex nbd



Suggests: (p,q) cusp $\overset{??}{\longleftrightarrow}$ negative end on $E(q,p)$

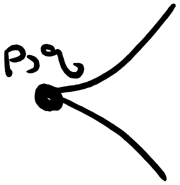
Rmk: the sizes of the relevant blowups are given by the (positive) continued fraction expansion of p/q.

Ex: $5/2 \leftrightarrow$ "weight seq" $\underbrace{2,2}_{2},\underbrace{1,1}_{2}$



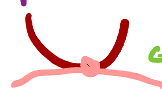
$\longrightarrow 5/2 = 2 + \frac{1}{2}$

§ From closed curves to punctured curves and back

Rmk: $\mathfrak{S}_{p,q}$  has well-defined tangent line

Cor: A negative end asymptotic to S^p in $\partial E(1, p/q+)$ is interchangeable with a $\mathfrak{S}_{p,q}$ singularity at a prescribed point, with some number $f(p,q)$ of prescribed derivatives

Compare: a negative end asymptotic to S^m in $\partial E_{sk} \approx \partial E(1, m+)$ is interchangeable with a point constraint, with $m-1$ prescribed derivatives.



$\langle \mathcal{T}_p^{m-1} \rangle$

Cor: $T_{p,q} = \#$ degree d $\left(= \frac{p+q}{3}\right)$ rational plane curves with a $\mathfrak{S}_{p,q}$ singularity at a prescribed point, with some number $f(p,q)$ of prescribed derivatives and $\delta = \frac{d^2 - pq + 1}{2}$ additional ordinary double points.

In particular, $T_{p,q} \neq 0 \iff \exists$ a degree d rational plane curve with a $\mathfrak{S}_{p,q}$ singularity

Ex (odd index Fibonacci curves): For $(p,q) = \frac{F_{i+2}}{F_{i-2}}$, i odd, ($\Rightarrow d = \frac{p+q}{3} = F_i$) have $\delta = 0$ and $T_{p,q} = 1$.

Orevkov (birational transformations), Cristofaro-Gardiner-Hind (ECH cobordism map)

Ex (even index Fibonacci curves): For $p/q = \frac{F_{i+2}}{F_{i-2}}$, $i \equiv 0 \pmod 4$,

$T_{p,q} > 0$ (Cristofaro-Gardiner-Hind-McDuff)

Corresponding curves have $\delta = 1$ double points.

Rmk: Can show $T_{p,q} \neq 0$ iff $\exists$ a positively immersed symplectic cobordism from $\Pi(p,q)$ to $\Pi(d,d)$

in $S^3 \times I$

transverse knot w/ maximal sl

d cpnt link

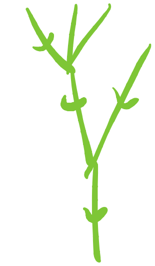
§ Tree formula:

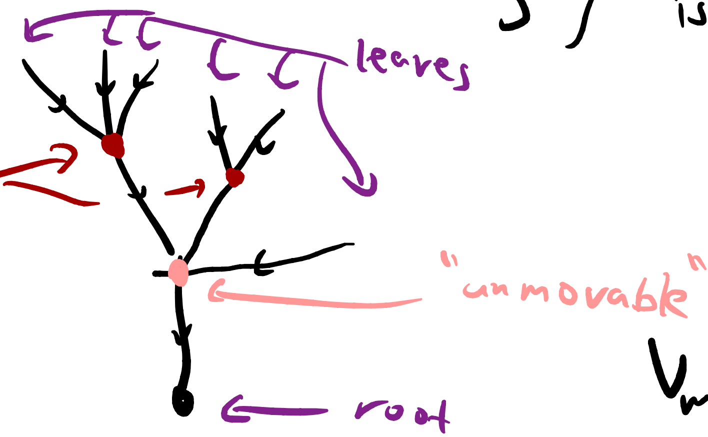
$\to$ For $x \in \mathbb{R}_{\geq 1}$, $k \in \mathbb{Z}_{\geq 1}$, put $(i_k^x, j_k^x) = \underset{i+j=k}{\operatorname{argmin}} \max(i, jx)$

Ex: $x = 1^+ \rightsquigarrow (i_k^x, j_k^x) = (\lceil k/2 \rceil, \lfloor k/2 \rfloor)$

Ex: $x \gg 1 \rightsquigarrow (i_k^x, j_k^x) = (k, 0)$

$$\mathcal{T}^d = \{\text{rooted trees with } d \text{ leaves}\}/\text{isomorphism}$$





$$V_{in}(T) = \{\text{internal vertices of } T\}$$

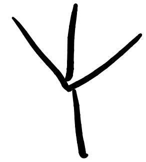
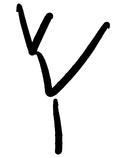
$$V_{mov}(T) = \{\text{movable vertices of } T\}$$

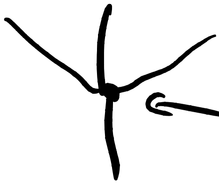
Def: For $v \in V_{in}(T)$, leaf number $\ell f(v)$ is the number of leaves above v.

Def: For $v \in V_{in}(T)$, put
$$\#^{p/q}(v) := \frac{L^{p/q}_{3\sum k_i - 1}!\; J^{p/q}_{3\sum k_i - 1}!}{\left(\sum L^{p/q}_{3k_i-1}\right)!\left(\sum J^{p/q}_{3k_i-1}\right)!}$$
where $k_1, \dots, k_n$ are the leaf numbers of the incoming vertices to v.

Thm (w/ G. Mikhalkin):
$$T_{p,q} = \frac{1}{p} 2^d \sum_{T \in \mathcal{T}^d} \frac{(-1)^{|V_{in}(T)|}}{|Aut(T)|} \left(\prod_{v \in V_{in}(T)} \#^{p/q}(v)\right)\left(\prod_{v \in V_{mov}(T)} \left(1 - \binom{2\ell f(v)}{\ell f(v)} 2^{-\ell f(v)}\right)\right)$$

$d = \frac{p+q}{3}$

Ex: $(p,q) = (8,1)$ $\qquad \mathcal{T}^3 = \{$  ,  $\}$

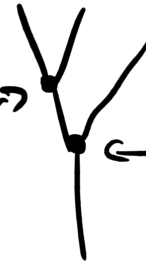


$\#(r) = \dfrac{8!}{6!}$

$\mathrm{lf}(r) = 3$

$|\mathrm{Aut}| = 3!$

$|V_{in}| = 1$



$\#(v_2) = \dfrac{8!}{7!}$ $\qquad \mathrm{lf}(v_2) = 3$

$|\mathrm{Aut}| = 2!$

$\#(v_1) = \dfrac{5!}{4!}$

$|V_{in}| = 2$

$\rightsquigarrow T_3 = T_{8,1} = 4$