From Floer to Hochschild via matrix factorisations

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Motivation

X compact toric variety, monotone

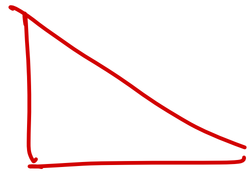
Δ moment polytope

Theorem (Batyrev, Givental)

Laurent poly defined combinatorially from Δ

$$QH^*(X) \cong \operatorname{Jac} W$$

Example $\quad X = \mathbb{CP}^2$

$\Delta =$  $\rightsquigarrow \quad W = x + y + \frac{1}{xy}$

$$QH^*(X) = \Bbbk[H] / (H^3 - 1)$$

$$\mathrm{Jac}\, W = \Bbbk[x^{\pm 1}, y^{\pm 1}] \Big/ \left(1 - \frac{1}{x^2 y},\ 1 - \frac{1}{x y^2}\right)$$

$$x = y, \quad x^3 = 1$$

Proof idea (Fukaya–Oh–Ohta–Ono, Biran–Cornea)

Consider monotone toric fibre $L \subset X$

$HF^*(L,L)$:= Floer cohomology of L

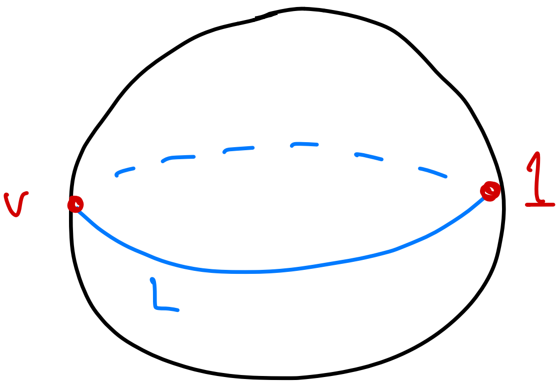
with "$\mathbb{K}[H_1(L;\mathbb{Z})]$ coefficients"

Fact $HF^*(L,L) \cong \mathrm{Jac}\, W$

Example

$$L = S^1_{eq} \subset X = S^2$$

$$\mathbb{k}[H_1] = \mathbb{k}[x^{\pm 1}]$$



$$d1 = v - v = 0$$

$$dv = x \cdot 1 - x^{-1} \cdot 1 = x \frac{\partial W}{\partial x} \cdot 1$$

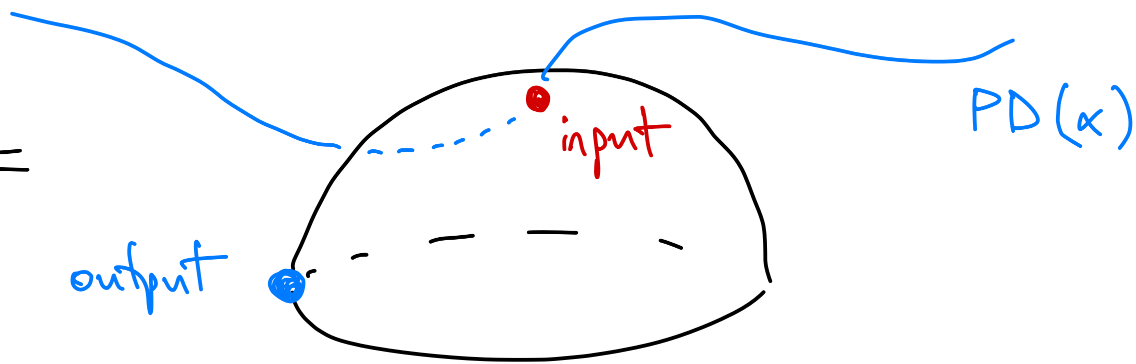
$$\mathcal{HF}^*(L, L) = \operatorname{Jac} W \cdot 1$$

Have unital algebra homomorphism

$$\widetilde{CO}^0 : QH^*(X) \longrightarrow \mathcal{HF}^*(L, L)$$

or KS or quantum module action

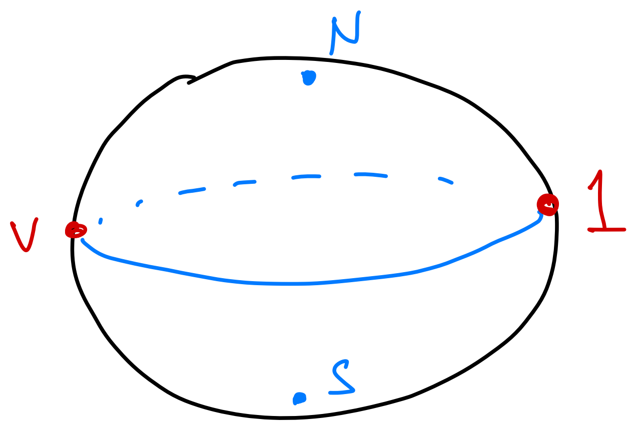
$$\widetilde{CO}^0(\alpha) =$$



$$QH^*(X) \xrightarrow{\widetilde{CO}^0} HF^*(L,L) \cong \mathrm{Jac}\, W$$

Some work $\rightsquigarrow$ this is an isomorphism

Example



$$\widetilde{CO}^0(N) = x \cdot 1$$

$$\widetilde{CO}^0(S) = x^{-1} \cdot 1$$

Second ingredient – generation criterion

Fukaya category $\mathcal{F}(X)_\lambda$

m_0 / curvature / obstruction number

$$QH^*(X) = \bigoplus_{\substack{\text{evals } \lambda \\ \text{of } c_1(X)}} QH^*(X)_\lambda$$

generalised λ-espace

Theorem (Abouzaid, Sheridan)

An object K in $\mathcal{F}(X)_\lambda$ split-generates if

$$\mathcal{CO}_K : QH^*(X)_\lambda \longrightarrow HH^*(CF^*(K,K))$$

is injective.

Remark The analogous result holds for finer decompositions of $QH^*(X)$ and $\mathcal{F}(X)$

Aside Have commuting unital maps

$$\begin{tikzcd} QH^*(X)_\lambda \arrow[r, "CO_K"] \arrow[dr, "CO_K^0"'] & HH^*(CF^*(K,K)) \arrow[d, "\text{projection to length zero}"] \\ & HF^*(K,K) \end{tikzcd}$$

So if $\dim QH^*(X) = 1$ then CO_K is injective

Theorem (Evans – Lekili)

For X compact monotone toric, each $\mathcal{F}(X)_\alpha$ is split-generated by L_α ← fibre L with some local system

Doesn't use generation criterion, but ...

Corollary (Evans – Lekili, I Smith)

$$QH^*(X)_\alpha \cong HH^*(\mathcal{F}(X)_\alpha)$$

Upshot Reasonable to guess that

$$CO_{L_\alpha} : QH^*(X)_\alpha \longrightarrow HH^*(CF^*(L_\alpha, L_\alpha))$$

is an isomorphism

Recall from earlier

$$\widetilde{CO}^{0} : QH^*(X) \longrightarrow HF^*(L, L) \cong \mathrm{Jac}\, W$$

is an isomorphism

<u>Question</u> Can we relate the two maps?

$$\widetilde{CO}^{\circ} : QH^*(X) \longrightarrow HF^*(L, L)$$

$$CO_{L_\alpha} : QH^*(X)_\alpha \longrightarrow HH^*(CF^*(L_\alpha, L_\alpha))$$

Answer Yes!

X any monotone symplectic manifold, compact or nice at infinity

L any monotone Lagrangian torus in X

more complicated statements for non-tori

Can define $\mathcal{HF}^*(L,L)$ as before

$H^*(\mathcal{CF}^*(L,L))$

Superpotential $W_L = \sum_{\substack{\text{index 2} \\ \text{discs } u}} z^{[\partial u]} \in \mathbb{K}[H_1(L;\mathbb{Z})]$

<u>Theorem</u> (Cho-Oh) For toric L $\quad W_L = W$

View W_L as a function

$$\{\text{local systems on } L\} \longrightarrow \mathbb{K}$$
$$\mathcal{L} \longmapsto \sum_u \mathrm{hol}_{\partial u}(\mathcal{L})$$

Fix a critical point $\mathcal{L}$ of W_L

Let $\mathbb{L} = (L, \mathcal{L}) \in \mathcal{F}(X)_{W_L(\mathcal{L})}$

Have A_∞-algebras $CF^*(\mathbb{L}, \mathbb{L})$ and

$$CC^*(CF^*(\mathbb{L}, \mathbb{L})) \leftarrow H^* \neq 0 \text{ since}$$

$\mathcal{L}$ a critical point

Theorem 1 There is a c-unital A_∞-algebra map

$$\Theta : \mathcal{CF}^*(L, L)^{op} \longrightarrow CC^*(CF^*(\mathbb{L}, \mathbb{L}))$$

Theorem 2 The following diagram commutes

$$\begin{tikzcd}
 & QH^*(X) \arrow[dl, "\widetilde{CO}^\circ"'] \arrow[dr, "CO_{\mathbb{L}}"] & \\
\mathcal{HF}^*(L,L)^{(op)} \arrow[rr, "H(\textcircled{H})"'] & & HH^*(CF^*(\mathbb{L},\mathbb{L}))
\end{tikzcd}$$

Remark In compact toric case, char 0, not necessarily monotone, FOOO construct

$$\begin{tikzcd}
& QH^*(X) \arrow[dl, "\hat{q} \text{ or } CO_{\mathbb{L}}"'] \arrow[dr, "ks \text{ or } \widetilde{CO}^{o}"] & \\
HH^*(CF^*(\mathbb{L}, \mathbb{L})) \arrow[rr, "k"'] & & \mathrm{Jac}\, W
\end{tikzcd}$$

Theorem 3 The map

$$\widehat{H(\Theta)} : \widehat{\mathcal{HF}^*(L,L)} \longrightarrow HH^*(CF^*(\mathbb{L},\mathbb{L}))$$

is (defined and) an isomorphism, where

$\widehat{\ }$ denotes completion at the maximal ideal

$\mathfrak{m}_{\mathcal{L}} \subset \mathbb{K}[H_1(L)]$ defining $\mathcal{L}$.

Example W_L has isolated critical points.

Then $HF^*(L, L) \cong \mathrm{Jac}\, W_L$

$$= \bigoplus_{\substack{\text{crit pts} \\ \alpha}} (\mathrm{Jac}\, W_L)_\alpha$$

And $\widehat{HF^*(L,L)} \cong (\mathrm{Jac}\, W_L)_\ell$

Example ω_L is constant

Then $\mathcal{HF}^*(L,L) = H^*(L) \otimes \Bbbk[H_1(L)]$

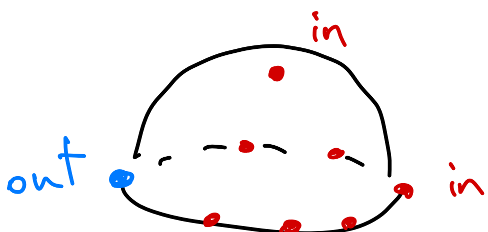
And $\widehat{\mathcal{HF}^*(L,L)} \cong H^*(L) \otimes \Bbbk[[t_1, \dots, t_n]]$

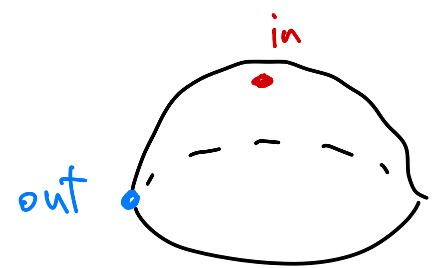
$$\cong \text{ HKR}$$

$HH^*(CF^*(\mathbb{L},\mathbb{L})) \cong HH^*(C^*(L))$

Consequences

- Fukaya-categorical interpretation of $\mathcal{HF}^*$
- Geometric interpretation of $HH^*(CF^*)$
- Makes $CO_{\amalg}$ computable via $\widetilde{CO}^\circ$





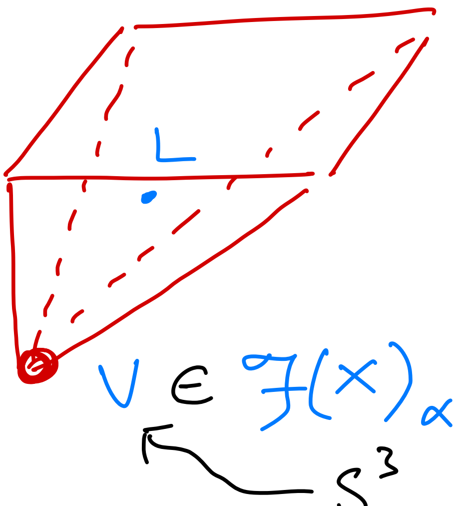
- New proof of toric generation result
- New generation results outside toric case

  eg Chekanov torus split-generates $\mathcal{F}(\mathbb{CP}^2)$ in characteristic 3

- Access to A_∞-operations on $QH^*(X)$ or $CC^*(\mathcal{F}(X))$ via those on $CF^*(L, L)$

Example X = Quadric 3-fold $\quad \mathrm{char}\,\mathbb{k} = 3$

$$QH^*(X) = \mathbb{k}[H]/(H^4 - 4H) \cong \underbrace{\mathbb{k}[H]/(H)}_{QH_\alpha} \oplus \underbrace{\mathbb{k}[H]/(H-1)^3}_{QH_\beta}$$



$$W_L = x + y + \frac{z}{x} + \frac{z}{y} + \frac{1}{z}$$

(Nishinou–Nohara–Ueda)

Can check

- $\operatorname{crit} W_L = \{ \mathcal{L} = (-1, -1, 1) \}$

- $\operatorname{Jac} W_L \cong \mathbb{k}[x^{\pm 1}, z^{\pm 1}] \big/ ((x+1)^3, z - x^2)$

- $CO^0 : QH^*_\beta \longrightarrow \operatorname{Jac} W_L$ is an isomorphism
  $H \longmapsto \frac{1}{z}$

So $(L, \mathcal{L})$ split-generates $\mathcal{F}(X)_\beta$

How to construct $\textcircled{H} : \mathcal{CF}^{op} \to CC^*(CF^*)$

Associated to $\mathbb{L}$ is a localised mirror functor (Cho-Hong-Lau)

$$\mathrm{LMF} : \mathcal{F}(X)_\lambda \longrightarrow mf(W_L - \lambda)$$

$W_L(\mathcal{L})$

$$\mathbb{L} \longmapsto \mathcal{E}$$

This gives an A_∞-algebra quasi-isomorphism

$$\Phi : \underset{\text{end}(\mathbb{L})}{CF^*(\mathbb{L},\mathbb{L})} \longrightarrow \underset{\text{end}(\mathcal{E})}{\mathcal{B}}$$

$\rightsquigarrow$ $\mathcal{B}$ is $CF^*(\mathbb{L},\mathbb{L})$-bimodule

So can define $CC^*(CF^*(\mathbb{L},\mathbb{L}), \mathcal{B})$

$$
\begin{tikzcd}
CC^*(CF^*(\mathbb{L},\mathbb{L})) \arrow[d, "\Phi_*"', "\wr"] & \\
CC^*(CF^*(\mathbb{L},\mathbb{L}), \mathbb{B}) & \mathcal{CF}^*(L,L)^{op} \arrow[l, "\Psi"'] \arrow[ul, dashed, "\Theta"'] \\
CC^*(\mathbb{B}) \arrow[u, "\Phi^*", "\wr"'] &
\end{tikzcd}
$$

computable !

A matrix factorisation of $W_L - \lambda \in \mathbb{K}[H_1(L)]$

$$\mathbb{K}[H_1(L)] = S$$

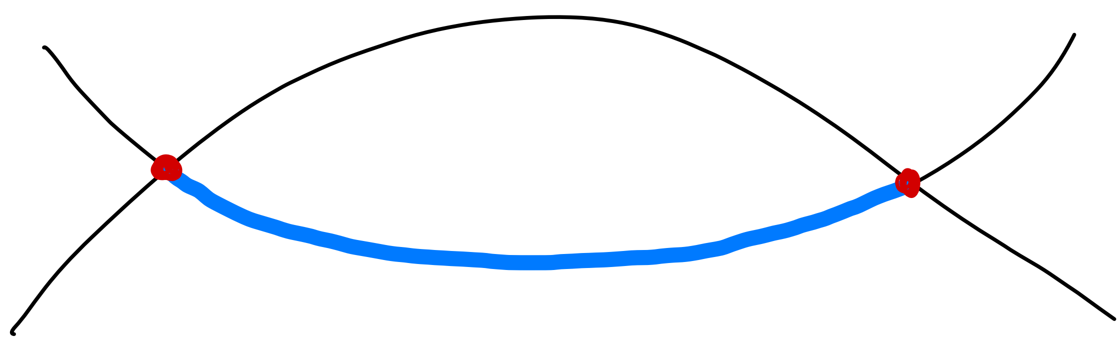
is a $\mathbb{Z}/2$-graded projective S-module M

with a twisted differential d satisfying

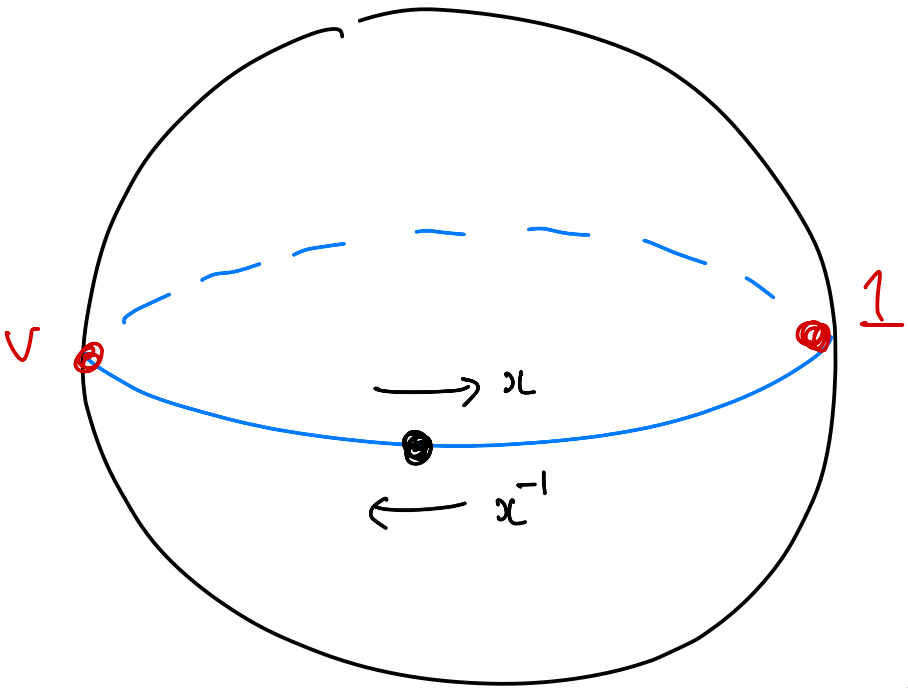
$$d^2 = (W_L - \lambda) \cdot \mathrm{id}_M$$

For us $\mathcal{E} = CF^*(L, L) \otimes S$

and d counts strips like d_{Floer} but weighted along the bottom edge



Example



$$d\,1 = x^{-1}v - v$$

$$dv = 1 - x\,1$$

$$d^2 = (x^{-1} - 1)(1 - x)$$

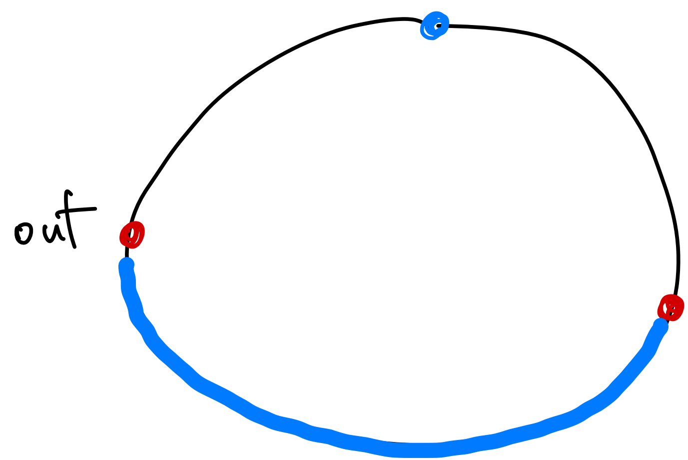
$$= \left(x + \frac{1}{x}\right) - 2$$

$$W_L \qquad \lambda$$

compare: Damian

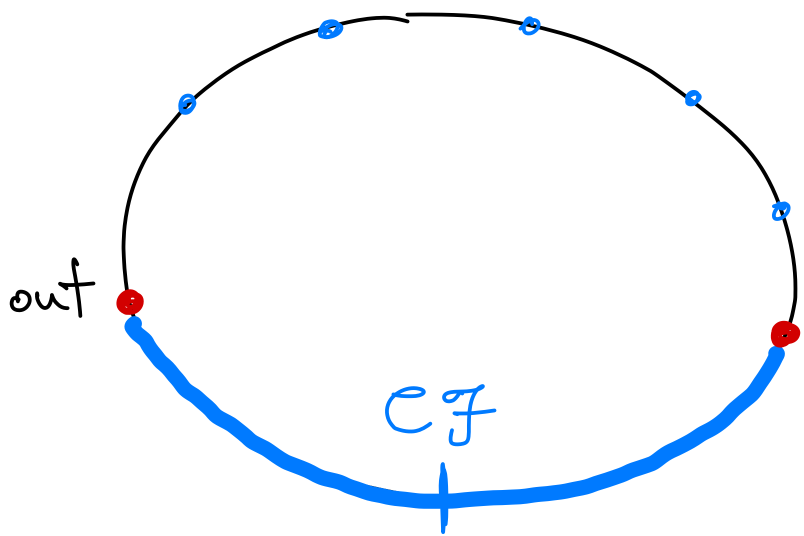
$$\Phi^1 : CF^*(\mathbb{L}, \mathbb{L}) \longrightarrow \mathcal{B}$$

$$CF^* \otimes \mathcal{E} \longrightarrow \mathcal{E}$$

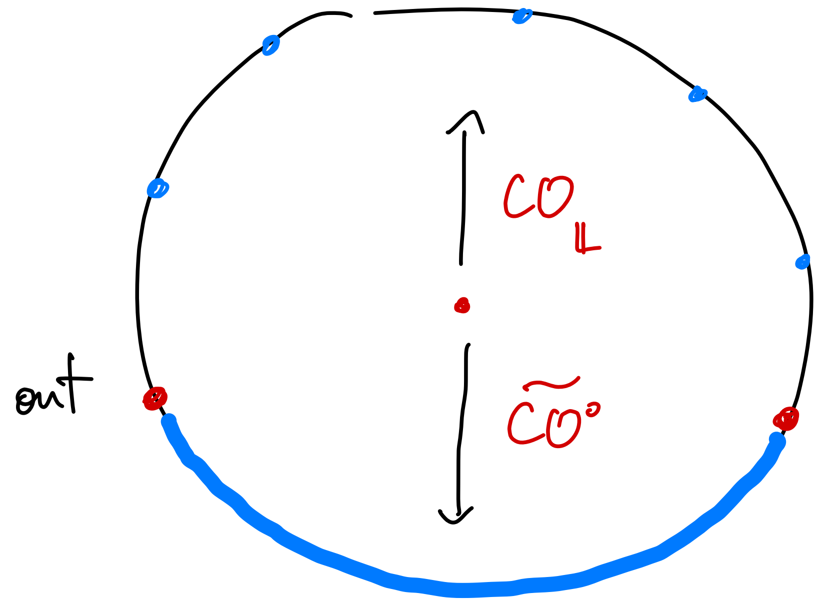


$$\mathcal{F}^1 : \mathcal{CF}^*(L, L)^{op} \longrightarrow CC^*(CF^*(\mathbb{L}, \mathbb{L}), \mathcal{B})$$

$$\mathcal{CF}^* \otimes (CF^*)^{\otimes k} \otimes \mathcal{E} \longrightarrow \mathcal{E}$$



Compatibility with $\mathcal{CO}$



Thanks for

listening!