Tate Homology and Powered Flybys

Kevin Ruck

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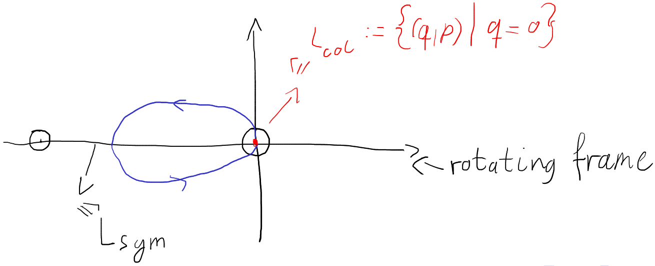
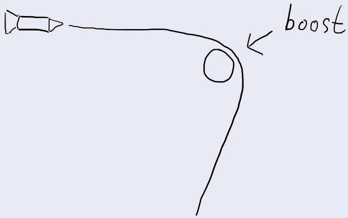
In the setting of the planar circular restricted three body problem there are infinitely many symmetric consecutive collision orbits for all energies below the first critical energy value, which intersect their symmetry axis on the straight line between the second and the main body.



Powered Flybys

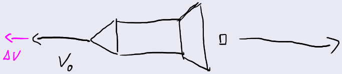
The Manoeuvre



Powered Flybys

The Rough Idea:

A rocket uses the conservation of momentum to gain velocity



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- $\rightarrow$ The gain of kinetic energy is big, if the starting velocity is high.
- $\rightarrow$ This phenomenon is called the Oberth effect after Hermann Oberth (1923)

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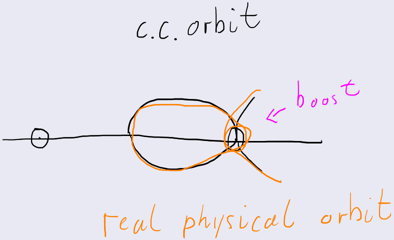

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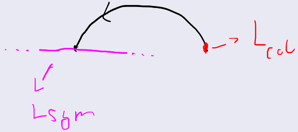
- For every c.c. orbit there is a flyby orbit close by.
- We have a sharp mathematical definition for c.c. orbits.



Orientation of C.C. Orbits

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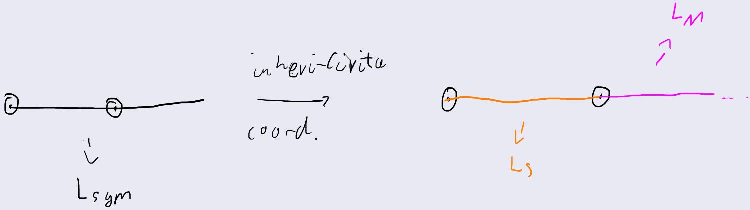
$$H \xrightarrow{\text{regularise}} |z|^2 \mathcal{L}^* H$$



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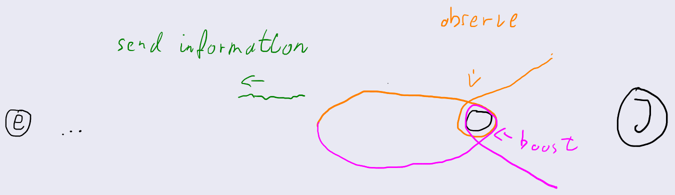
Why Orientation matters:



A Short Introduction to Lagrangian RFH

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on a manifold $P(M, L) \times \mathbb{R}$, where

$$P(M, L) := \{x \text{ path in } M \text{ with } x(0) \in L,\ x(1) \in L\}$$

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The Lagrangian Rabinowitz Floer Homology is then defined as

$$RFH(M, H, L) := \frac{\ker \partial}{\mathrm{im} \partial}.$$

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The powerfull property of RFH is that it only depends on the Hamiltonian system up to homotopy.

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Solution:

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G-Equivariant Lagrangian RFH

The idea of equivariant RFH is that dividing out a symmetry will improve the strength of the resulting homology.

$$CRF \xrightarrow{/G} CRF^{\underline{G}}$$ → finite group, acts free

$$\partial \xrightarrow{/G} \partial^G$$ ← count only up to G-action

$$\Rightarrow \quad RFH^G = \frac{\mathrm{ker}\partial^G}{\mathrm{im}\partial^G}$$

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$$RFH_*^G(M, H, L) = TH_*(G, \mathbb{Z}_2).$$

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5. Remember that $TH_i(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \ \forall i \in \mathbb{Z}$ $\square$