Γ-support, γ-coisotropic sets and Applications

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1 Spaces and metrics

$\mathcal{L}(M, d\lambda) = \{$ exact, closed Lagrangians in $(M, d\lambda)\}$

$\mathcal{L}_0(M, d\lambda) = \{ L \in \mathcal{L}(M, d\lambda) \mid L = \varphi_H(L_0) \}$

φ_H Ham flow of H.

Spectral metric γ

- using g. functions (V. 92) $\mathcal{L}_0(T^*N)$
- using Floer homology [Leclercq '08, Leclercq - Zapolsky '18) on $\mathcal{L}_0(M, d\lambda)$
- using sheaves (Viterbo '14) in $\mathcal{L}(T^*N)$

They are all equal (V'95, V'19)

Properties

1) it is a metric!

2) it is invariant by Ham maps

$$\gamma(\varphi_H(L_1), \varphi_H(L_2)) = \gamma(L_1, L_2)$$

3) Continuous w.r. Hofer metric

$$\gamma(\varphi_H(L), L) \leq \|H\|_{C^0}$$

4) $$\gamma(gr(df), O_N) = osc(f) \leq 2\|f\|_{C^0}$$

$$gr(df) = \{(x, df(x)) \mid x \in N\}$$

N closed mfd.

Rks. (1) The same can be done on $DHam(M, \omega)$ (Schwarz '00, Oh '08).

(2) $(\mathcal{L}, \gamma)$ is not complete

example. $f_n \xrightarrow[C^0]{} f$ $\quad f \in C^0$

$$\gamma(gr(df_n), gr(df_m)) \leq 2 \, \| f_n - f_m \|_{C^0}$$

Q: (Oh, Stecconi): is $(\mathcal{L}, \gamma)$ a Baire space

A No, $(\mathcal{L}, \gamma)$ is not even a Baire space.

2. The Humilière completion (V. Humi '08)

Def: Denote by $(\widehat{\mathcal{L}}, \gamma)$ and $(\widehat{DHam}, \gamma)$ the completions of the above spaces.

Goal: Understand $(\widehat{\mathcal{L}}, \gamma)$ $(\widehat{DHam}, \gamma)$.

Rk. Elts of $(\widehat{\mathcal{L}}, \gamma)$ $\cap$ $(\widehat{DHam}, \gamma)$ occur quite naturally.

Ex: 1) $f \in C^0(N, \mathbb{R})$ $gr(df) \in \widehat{\mathcal{L}}$

2) $\widehat{DHam}$ occurs naturally when doing symplectic homogenization.

start $H \in C^\infty(T^*T^n, \mathbb{R})$

$H_k(q,p) = H(kq,p)$ $k \in \mathbb{N}$ $k \to +\infty$.

the limit is described by $\bar{\varphi} \in DHam$

3) $H \in C^0(M, \mathbb{R})$ then what is the meaning of φ_H? Sometimes one can give meaning to φ_H in DHam.

Rk: Bad news: elements of $\hat{\mathcal{L}}(M, d\lambda)$ are not subsets of M, elts of DHam$(M, d\lambda)$ do not act on M.

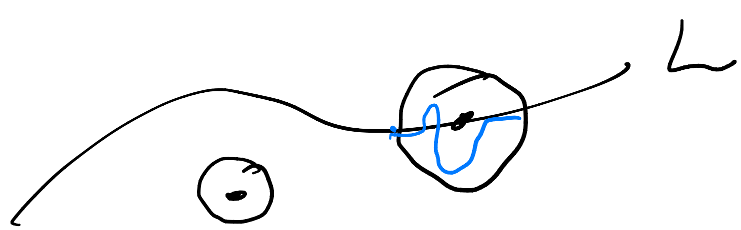
Good news. $\widehat{DHam}(M, d\lambda)$ acts by isometries on $\hat{\mathcal{L}}(M, d\lambda)$.

3 The γ-support for elts in $\hat{\mathcal{L}}$

Def Let $L \in \hat{\mathcal{L}}(M, d\lambda)$. We say that $x \in M$ is in the γ-support of L if

$\forall U \ni x \ \exists \varphi \in DHam_c(U)$ s.t.

$$\gamma(\varphi L, L) > 0 \quad (\Rightarrow \varphi L \neq L!)$$



Properties: 1) If L is smooth $\gamma\text{-supp}(L) = L$.

2) This is invariant by DHam$(M, d\lambda)$

$$\gamma\text{-supp}(\varphi(L)) = \varphi\, \gamma\text{-supp}(L).$$

3) If $L_k \xrightarrow[\gamma]{} L$ we have

$$\gamma\text{-supp}(L) \subset \liminf_{\text{Hausd}} \gamma\text{-supp}(L_k).$$

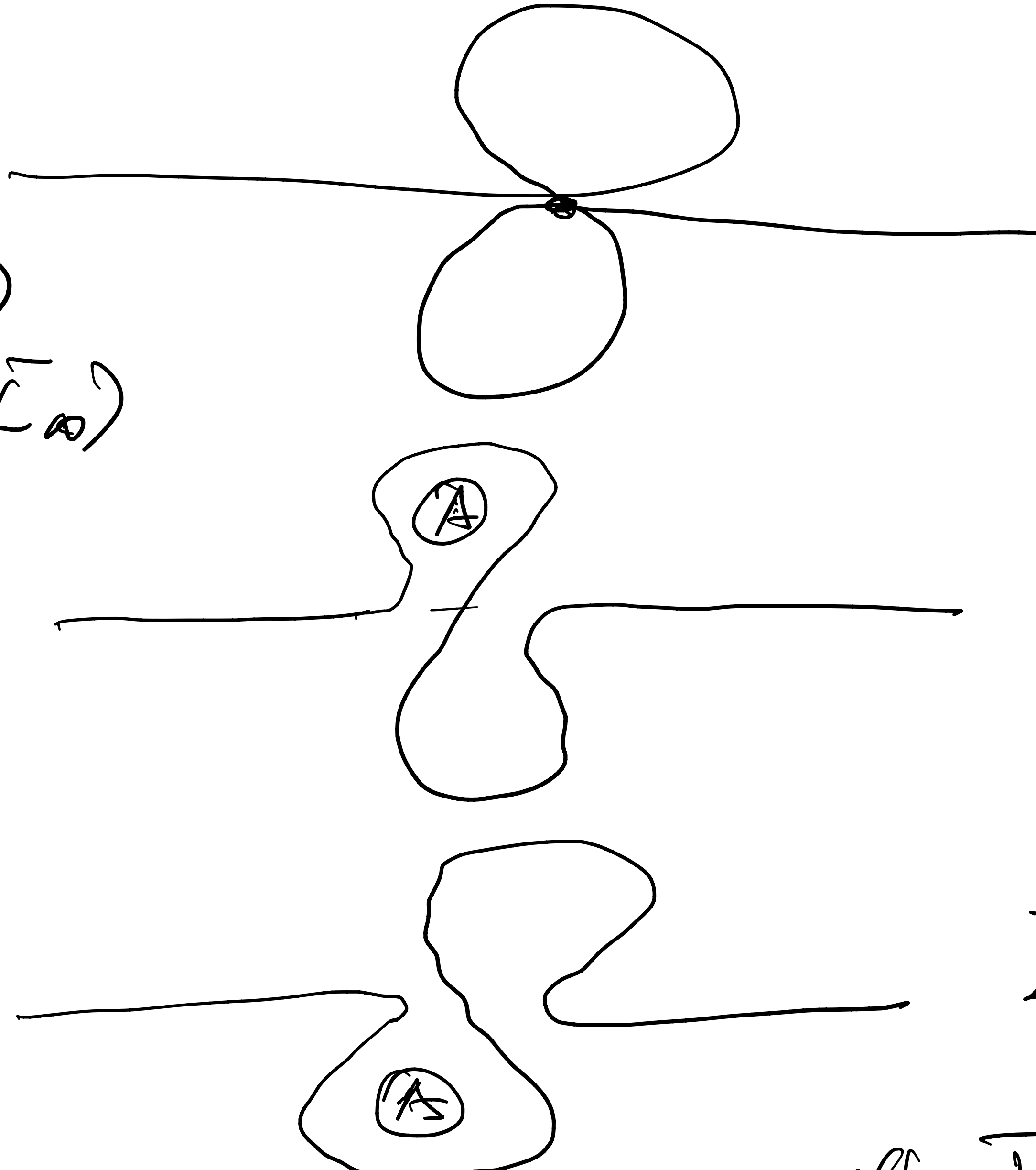
Goal. Understand $\gamma\text{-supp}(L)$

e.g. If $V = \gamma\text{-supp}(L_0)$ how many $L \in \hat{\mathcal{L}}$ have the same γ-support?

Example.

$V = \gamma\text{-supp}(L_\infty) = \gamma\text{-supp}(\bar{L}_\infty)$

T^*S^1



L_k

$L_k \to L_\infty$

$\bar{L}_k$

$\bar{L}_k \to \bar{L}_\infty$

$\gamma(L_k, \bar{L}_k) \simeq 2A$

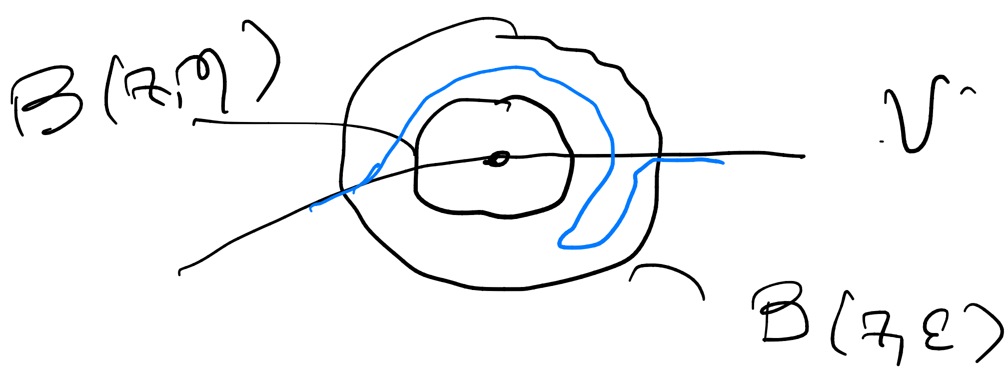
4 γ-isotropic sets:

(see M. Usher locally rigid '18).

Def Let $V \subset (M,\omega)$ and $z \in V$. We say that V is γ-coisotropic at z if:

$\forall\, 0 < \eta < \varepsilon$ $\exists\, \delta > 0$ s.t. for $\varphi \in DHam_c(B(z,\varepsilon))$

and $\varphi(V) \cap B(z,\eta) = \emptyset \Rightarrow \gamma(\varphi) \geq \delta$.

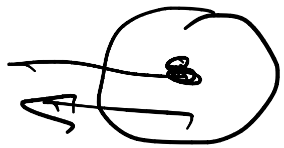


Properties This is invariant by homeomorphisms preserving γ (hence by sympl. homeomorphisms)

Examples. (1) If is smooth, closed submf^d then V is γ-coisotropic $\Leftrightarrow$ V is coisotropic

(2) $\{0\} \subset \mathbb{R}^2$ is not γ-coisotropic, and if S is codim ≥ 2 and symplectic, it is not γ-coisotropic.

(3) $[0,1] \subset \mathbb{R}^2$: it is γ-coisotropic at z $\Leftrightarrow$ $z \in \,]0,1[$



Thm (1) Let $L \in \hat{\mathcal{L}}(M, d\lambda)$ then $\gamma\text{-supp}(L)$ is γ-coisotropic

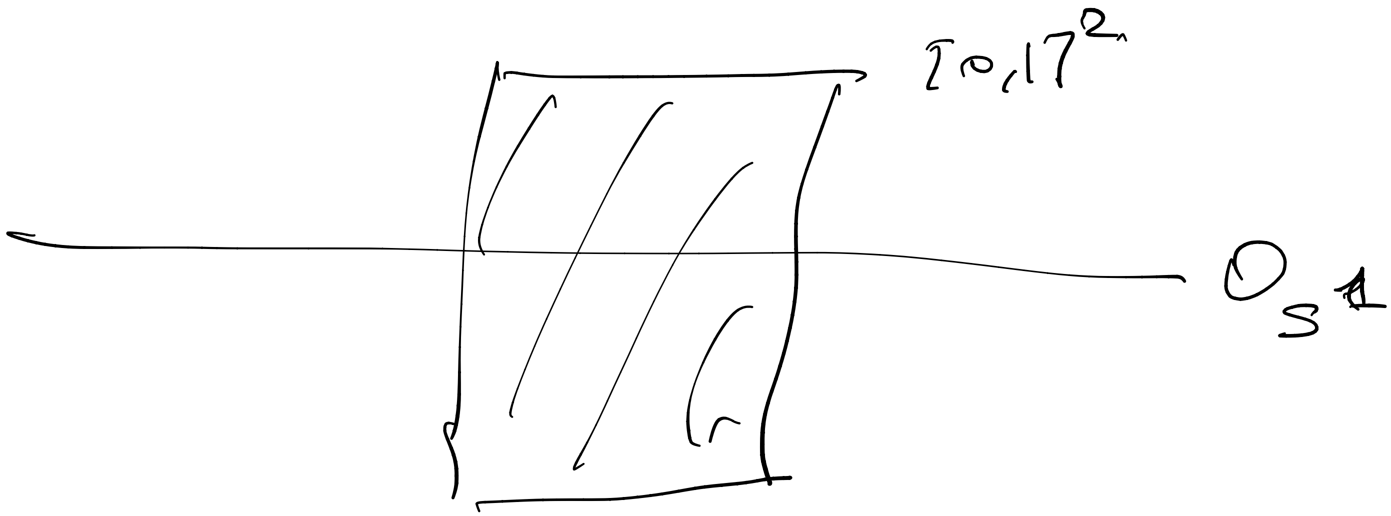
(2) (Peano Lagrangians) $\exists\, L \in \hat{\mathcal{L}}(T^*N)$ s.t.

$$\gamma\text{-supp}(L) = 0_N \cup [0,1]^n \times [0,1]^r \quad 0 \le r \le n$$

$\dim(N) = n$.

Rk. For (2) there are infinitely many possible L. (for $r > 0$).

$\underline{n=1}$



<u>Q</u>: For which V does the set of $L \in \hat{\mathcal{L}}$ s.t. $\gamma\text{-supp}(L) = V$ a compact set in $\hat{\mathcal{L}}$?

5 Regular Lagrangians

What happens when $\gamma\text{-supp}(L) = V$ is (exact) a Lagrangian submanifold (smooth).

Q: Is it true that $L = \widehat{V}$

$$\begin{array}{ccc} L & = & \widehat{V} \\ \nearrow & & \uparrow \\ \widehat{\mathcal{L}} & \longleftarrow & \mathcal{L} \end{array}$$

Thm Let N be a compact homogeneous space and assume $\gamma\text{-supp}(L) = V$ is in $\mathcal{L}_0(T^*N)$. Then

$$V = \gamma\text{-supp}(L) = L.$$

Consequences: what is the center of $\widehat{DHam}$?

Thm $\Rightarrow$ $\widehat{DHam}(T^*T^n)$ the center is trivial.

6 Other applications

(A) Singular Hamiltonians.

e.g. $H \in C^\infty(M, V)$ does H define an element $\varphi_H \in DHam(M)$

(V: closed subset)

Def. We say that $V \subset M$ is not f-coisotropic at $z \in V$ if the map ($U \ni z$)

$$DHam(M,\omega) \longrightarrow DHam(M, \overline{(U \cap V)})$$

is onto

Thm: If V is nowhere γ-coisotropic then it is nowhere f-coisotropic.

Cor (V. Humilière) if $\dim V < n$ and V is smooth then any $H \in C^0(M \setminus V)$ defines an element in $\widehat{DHam}(M,\omega)$.

(B) (joint with S. Guillermou).

Let $\mathcal{F} \in D^b(N)$ then it has Singular support $SS(\mathcal{F})$. (defined by Kashiwara-Schapira)

SS(F) is "involutive": condition on the tangent cones of a set

<u>Thm</u> (Guillermou - V.):

SS(F) is γ-coisotropic.

<u>Rk</u> γ-coisotropic ⇒ involutive.

(C) (joint work with V. Humilière)

Let $\varphi : (M,\omega) \circlearrowleft$ dissipative / conformally symplectic: $\varphi^*\omega = c\,\omega$ $\quad 0<c<1$.

φ induces a map $\widehat{\mathcal{L}}(M, d\lambda) \longrightarrow \widehat{\mathcal{L}}(M, d\lambda)$

γ-contraction. (M.C. Arnaud & V. Humilière)

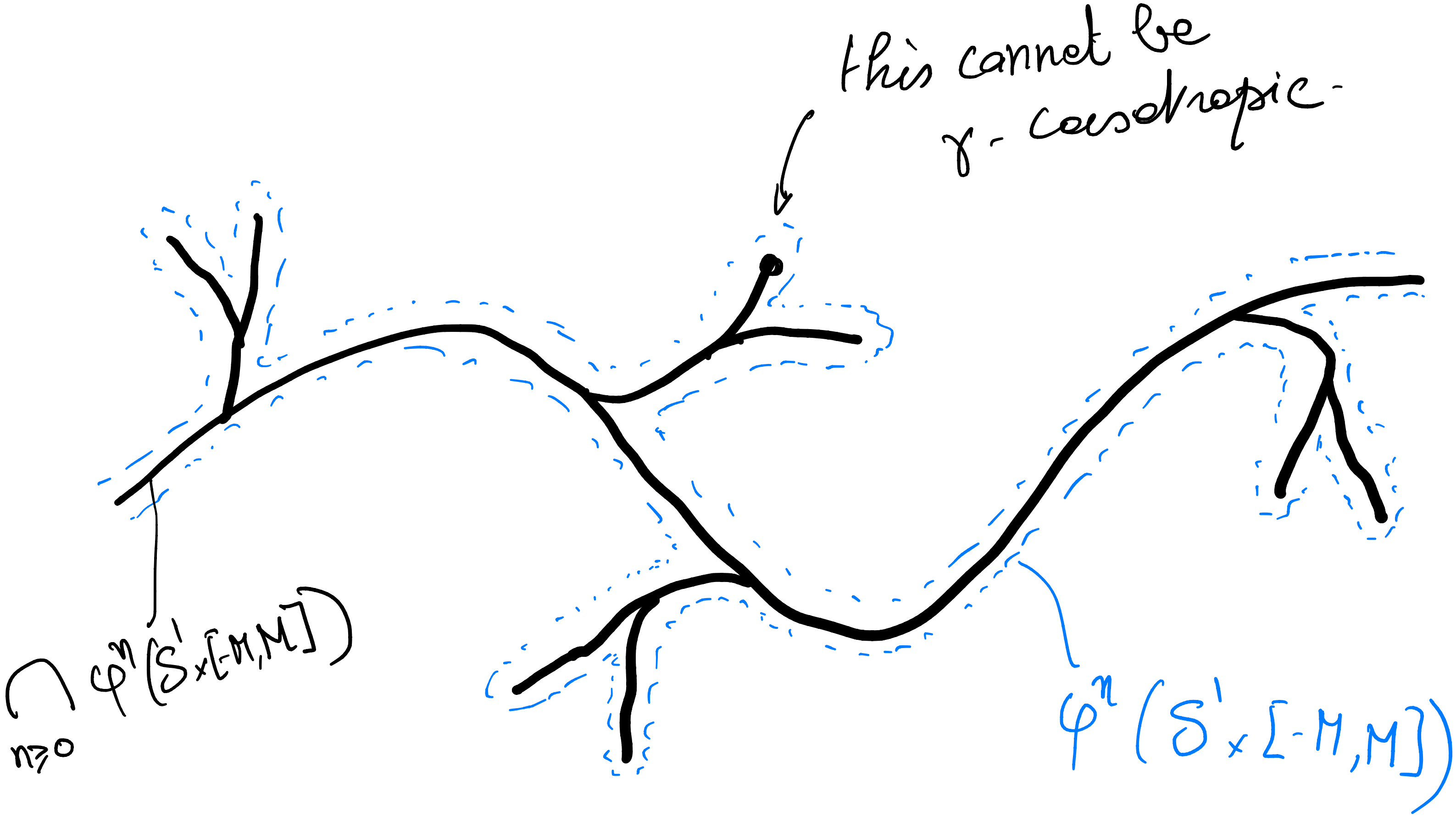
It has a unique fixed pt in $\widehat{\mathcal{L}}(M, d\lambda)$: $L_\infty(\varphi)$

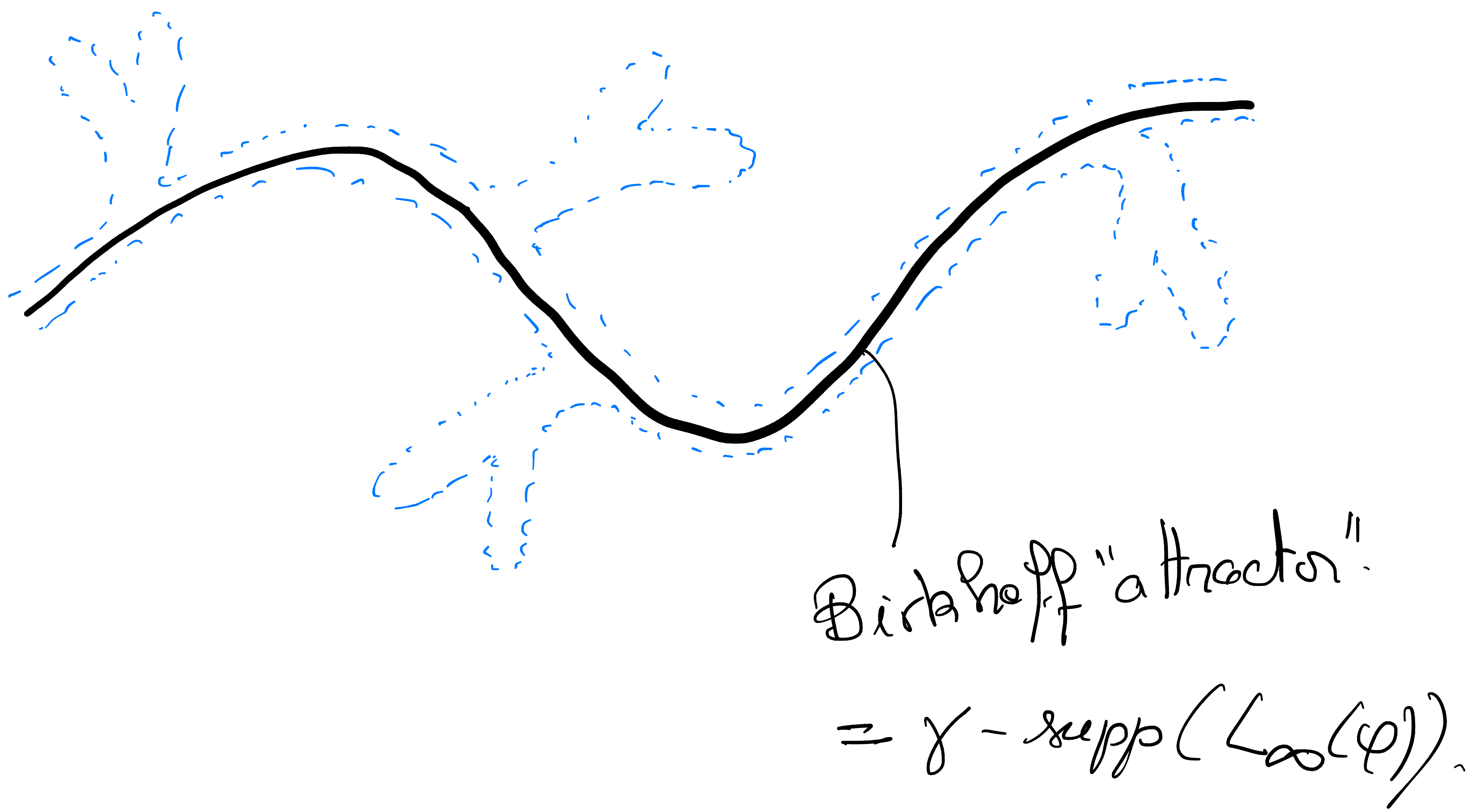
$\Rightarrow$ φ preserves $\gamma\text{-supp}(L_\infty(\varphi)) = V_\infty(\varphi)$.

We thus get an invariant set for φ !

<u>$n = 1$.</u> (T^*S^1) (Birkhoff '32, Charpentier '34, Le Calvez 86-91).

$\varphi : S^1 \times [-M, M] \circlearrowleft$ We have an invariant set : $\bigcap \varphi^n(S^1 \times [-M, M])$


This cannot be γ-coisotropic.
$\bigcap_{n\geq 0} \varphi^n(S^1\times[-M,M])$
$\varphi^n(S^1\times[-M,M])$


Birkhoff "attractor"
$= \gamma\text{-supp}(L_\infty(\varphi))$.