Viterbo's conjecture for Lagrangian products in $\mathbb{R}^4$

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Outline

Motivation

Main Results

Proof ideas

Viterbo's conjecture for the EHZ-capacity of convex sets

Conjecture (Viterbo's conjecture for the EHZ-capacity)

$$vol(C) \geq \frac{c_{EHZ}(C)^n}{n!}, \quad C \subset \mathbb{R}^{2n} \text{ convex}$$

$$c_{\mathrm{EHZ}}(C) = \min\{\mathbb{A}(\gamma) : \gamma \text{ closed characteristic on } \partial C\}$$

γ closed characteristic on ∂C: $\dot{\gamma}(t) \in JN_C(\gamma(t))$ a.e. on $\mathbb{R}/\mathbb{Z}$

Conjecture (Viterbo's conjecture as systolic inequality)

$$sys(C) \leq sys(B^{2n}) = 1, \quad C \subset \mathbb{R}^{2n} \text{ convex}$$

Systolic ratio: $\mathrm{sys}(C) = \dfrac{c_{\mathrm{EHZ}}(C)}{(n!\,\mathrm{vol}(C))^{\frac{1}{n}}}$

Related conjectures

Conjecture

All symplectic capacities coincide on the class of convex sets in $\mathbb{R}^{2n}$.

Coincidence of capacities $\Rightarrow$ Viterbo's conjecture for c_{EHZ} true

Remark: The **following implication** stated in the talk is **wrong**, since, among other reasons, having an equality case of Viterbo's conjecture for the EHZ-capacity which is not symplectomorphic to a Euclidean would not imply that the Gromov width (which is defined with "sup") of this equality case is less than its EHZ-capacity:

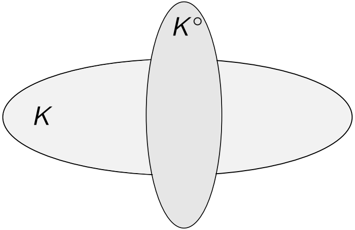
"Coincidence of capacities $\Rightarrow$ equ. cases symplectomorphic to Eucl. ball"

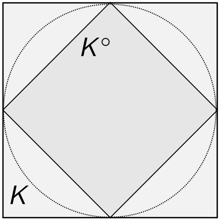
$$\text{Gromov width: } w_G(C) = \sup\left\{\pi r^2 : B_r^{2n} \overset{\text{sympl.}}{\hookrightarrow} C\right\}$$

Related conjectures

Conjecture (Mahler, 1939)

$$\nu(K) = \mathit{vol}(K)\,\mathit{vol}(K^\circ) \geq \frac{4^n}{n!}, \quad K \subset \mathbb{R}^n \textit{ centr. symm. convex body}$$





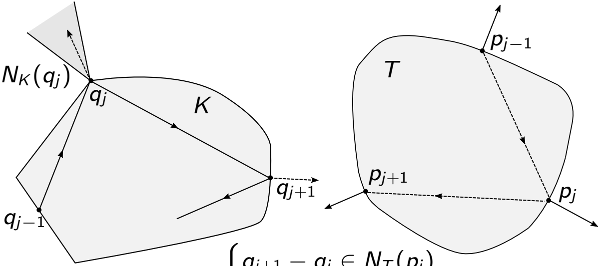
$$\nu(B^n) = \nu(\mathcal{E}^n) \overset{Bl.-S.}{\geq} \nu(K) = \mathrm{vol}\,(K \times K^\circ) \begin{cases} \overset{\text{Mahl. conj.}}{\geq} \frac{4^n}{n!} = \nu(\Box^n) = \nu(\Diamond^n) \\ \overset{\text{Vit. conj.}}{\geq} \frac{c_{\mathrm{EHZ}}\,(K \times K^\circ)^n}{n!} \overset{\text{A.A.-K.-O.}}{=} \frac{4^n}{n!} \end{cases}$$

Viterbo's conjecture $\Rightarrow$ Mahler's conjecture

The case of Lagrangian products

Theorem (R., Minkowski billiard char. of $c_{\mathrm{EHZ}}(K \times T)$)

$$c_{EHZ}(K \times T) = \min_{q \text{ cl. } (K,T)\text{-Mink. bill. traj.}} \ell_T(q), \quad K, T \subset \mathbb{R}^n \text{ convex}$$



$$\begin{cases} q_{j+1} - q_j \in N_T(p_j) \\ p_{j+1} - p_j \in -N_K(q_{j+1}) \end{cases}$$

Conjecture (Viterbo as a systolic Minkowski billiard inequality)

$$\min_{q \text{ cl. } (K,T)\text{-Mink. bill. traj.}} \ell_T^n(q) \leq n!\, vol(K), \quad K, T \subset \mathbb{R}^n \text{ convex}$$

Relevant Questions

Question 1:

Viterbo's conjecture for the EHZ-capacity of Lagrangian products true?
For which Lagrangian products?

Question 2:

What are equality cases of Viterbo's conjecture?
Classification of equality cases?

Question 3:

Are the equality cases symplectomorphic to Euclidean balls?

Question 4:

Characterization of equality cases?
Zoll-property?

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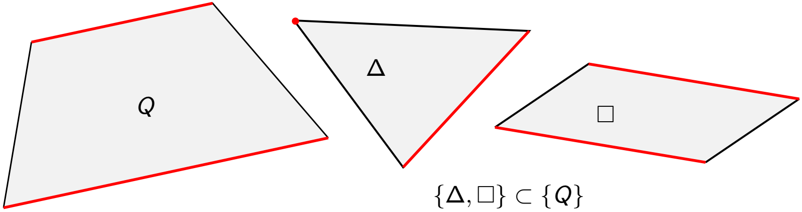
Answer to Question 1

Theorem (R.)

Let Q be any trapezoid in $\mathbb{R}^2$. Then, Viterbo's conjecture is true for all Lagrangian products

$$Q \times T,$$

where T is any convex body in $\mathbb{R}^2$.



$$\{\Delta, \Box\} \subset \{Q\}$$

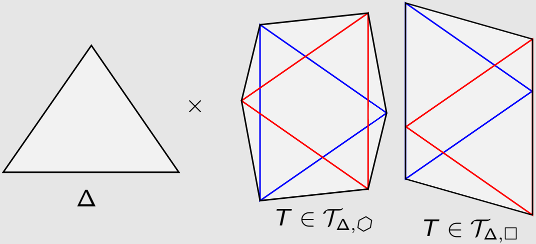
Corollary (Systolic Minkowski billiard inequalities for geometries having trapezoids as unit ball)

$$\min_{q \text{ cl. } (K,Q)\text{-Mink. bill. traj.}} \ell_Q^2(q) \leq 2\, vol(K), \quad Q \text{ trapezoid}$$

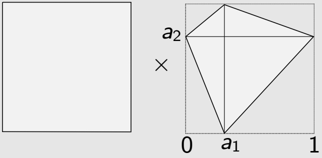
Answer to Question 2

Theorem (R., Classification of eq. cases for the trapez.-config.)

(i) $\Delta \times T$ *for any triangle* Δ, *any* $T \in \mathcal{T}_{\Delta,\square} \cup \mathcal{T}_{\Delta,\hexagon}$;



(ii) $\square \times \diamondsuit(a_1, a_2)$ *for any standard square* $\square$, *any diamond* $\diamondsuit(a_1, a_2)$;

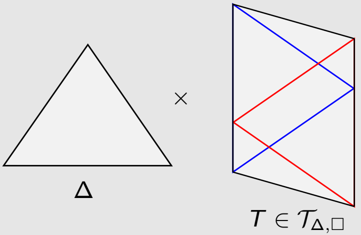


(iii) $Q \times P$ *for any "strict" trapezoid* Q, *cert. parallelogram* $P = P(Q)$.

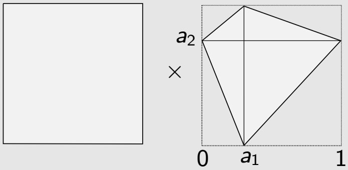
Answer to Question 3

Theorem (R., Eucl. balls from the symplectic point of view)

(i) $\mathring{\Delta} \times \mathring{T}$ *for any triangle* Δ, *any* $T \in \mathcal{T}_{\Delta,\Box}$;

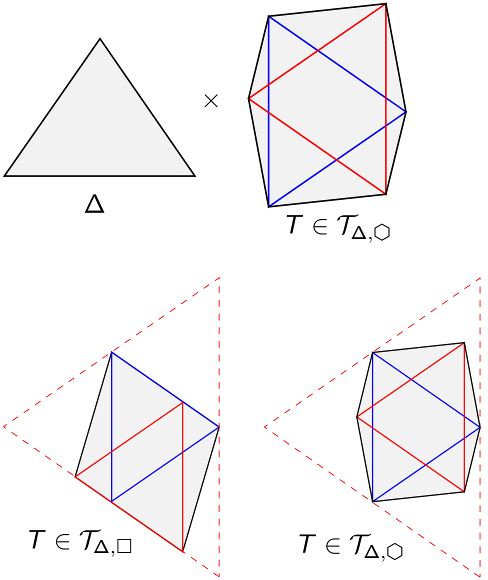


(ii) $\mathring{\Box} \times \mathring{\Diamond}(a_1, a_2)$ *for any standard square* $\Box$, *any diamond* $\Diamond(a_1, a_2)$;



(iii) $\mathring{Q} \times \mathring{P}$ *for any "strict" trapezoid* Q, *cert. parallelogram* $P = P(Q)$.

Open Question: Euclidean balls through symplectic glasses?



[Ostrover/Ramos/Sepe: (equil. triangle)×(regular hexagon) Eucl. ball]

Answer to Question 4

Theorem (R., Zoll-property)

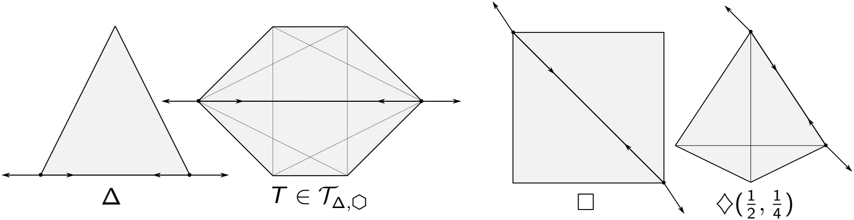
All equality cases $K \times T$ presented above (which are polytopes) satisfy the following Zoll-property:

(i) *every characteristic on $\partial(K \times T)$ that runs on the interiors of the facets almost everywhere*

- *is closed,*
- *runs over exactly 8 facets,*
- *minimizes the action;*

(ii) *the union of these characteristics is dense on $\partial(K \times T)$.*

The action-minimizing-property in (i) is not true in general:



Δ $\quad T \in \mathcal{T}_{\Delta,\hexagon}$ $\quad \square$ $\quad \diamondsuit(\frac{1}{2}, \frac{1}{4})$

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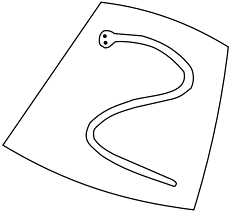
Answers to Questions 1 & 2: Proof Idea

Theorem (R., Viterbo's conjecture as a covering problem)

For convex $K, T \subset \mathbb{R}^n$:

$$\min_{q\text{ cl. }(K,T)\text{-Mink. bill. traj.}} \ell_T^n(q) \begin{cases} \leq n!\, vol(K) \\ = n!\, vol(K) \end{cases}$$

$$\Leftrightarrow \quad \min_{K \text{ covers any cl. curve } q \text{ with } \ell_T(q)=\sqrt{n!}} vol(K) \begin{cases} \geq 1 \\ = 1 \end{cases}$$



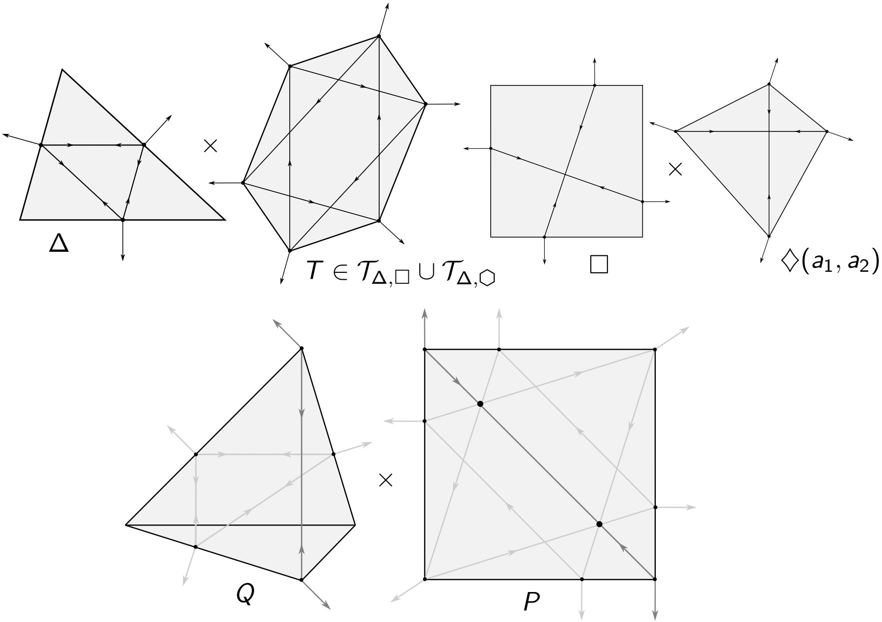
Related problems in geometry:

Moser's worm problem (1966)

Wetzel's problem (1973)

Bellman's lost-in-a-forest problem (1955)

Answers to Questions 1 & 2: Proof Idea



Answer to Question 3: Proof Idea

Theorem (R., Euclidean balls from the symplectic point of view)

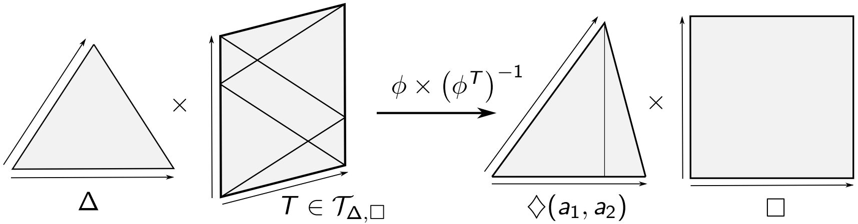
(i) $\mathring{\Delta} \times \mathring{T}$ *for any triangle* Δ, *any* $T \in \mathcal{T}_{\Delta,\square}$;

(ii) $\mathring{\square} \times \mathring{\diamondsuit}(a_1, a_2)$ *for any standard square* $\square$, *any diamond* $\diamondsuit(a_1, a_2)$;

(iii) $\mathring{Q} \times \mathring{P}$ *for any "strict" trapezoid* Q, *cert. parallelogram* $P = P(Q)$.

Theorem (Generalization of Schlenk's embedding result)

$$\mathring{\diamondsuit}(a_1, a_2) \times \mathring{\square} \overset{\text{sympl.}}{\cong} B^4_{\frac{1}{\sqrt{\pi}}}, \quad a_1, a_2 \in [0, 1]$$



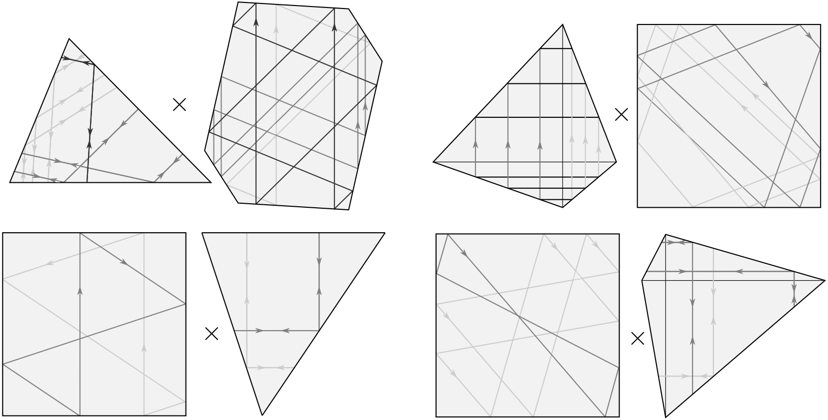
Answer to Question 4: Proof Idea

Theorem (R., Zoll property)

All equality cases $K \times T$ presented above (which are polytopes) satisfy:

(i) *every char. on $\partial(K \times T)$ that runs on the interiors of the facets a. e. is closed, runs over exactly 8 facets, and minimizes the action;*

(ii) *the union of these characteristics is dense on $\partial(K \times T)$.*

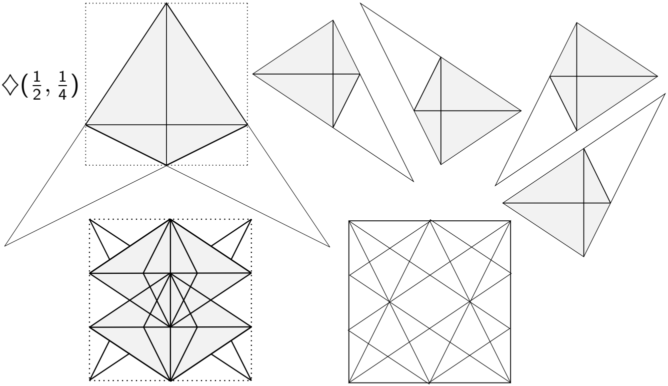


Thank you for the attention!

Questions?

Outlook

- **Other Lagrangian products in $\mathbb{R}^2 \times \mathbb{R}^2$**
  - (Quadrilateral)$\times T$: leads to a Euclidean covering-problem; Example:



  - Algorithm implemented for computing $c_{EHZ}(K \times T)$ for polytopes $K, T \subset \mathbb{R}^2$
- **Higher dimensions:** much more difficult to find length-minimizing Minkowski billiard trajectories, since

$$\text{dimension} \sim \#\{\text{billiard reflection points}\}$$

Viterbo's conjecture as covering problem in relation to famous problems in geometry

Worm problems in general: Given a collection $\mathcal{F}$ of n-dimensional figures F and a transitive group $\mathcal{M}$ of motions m on $\mathbb{R}^n$, find the minimal convex target sets $K \subset \mathbb{R}^n$–minimal in the sense of having least volume, surface volume, or whatever–so that for each $F \in \mathcal{F}$ there is a motion $m \in \mathcal{M}$ with $m(F) \subseteq K$.

Moser's worm problem: "Find a/the (convex) set of least area that contains a congruent copy of each arc in the plane of lenth one."
Setting: $\mathcal{F} =$ set of arcs in $\mathbb{R}^2$ of Euclidean length 1, $\mathcal{M}$: group of congruence transformations, Minimization: minimal area of the target sets $K \subset \mathbb{R}^2$

Wetzel's problem:
Setting: $\mathcal{F} =$ set of closed curves in $\mathbb{R}^2$ of Euclidean length α, $\mathcal{M}$: group of translations, Minimization: minimal volume of the target sets $K \subset \mathbb{R}^2$

Viterbo's conjecture for $K \times T$:
Setting: $\mathcal{F} =$ set of closed curves in $\mathbb{R}^n$ of ℓ_T-length α, $\mathcal{M}$: group of translations, Minimization: minimal volume of the target sets $K \subset \mathbb{R}^n$

Bellman's lost-in-a-forest problem: A hiker is lost in a forest whose shape and dimensions are precisely known to him. What is the best path for him to follow to escape from the forest? ($\Leftrightarrow$ **Moser's worm problem**)

Viterbo's conjecture as escape / lost-in-a-forest problem

Story: "Two hikers walk in a forest. One of them gets injured and is in need of medical attention. The unharmed hiker would like to make the emergency call. Although he has his cell phone with him, there is only reception outside the forest. He has a map of the forest, i.e., the shape of the forest and its dimensions are known to him, and a compass to orient himself in terms of direction. Furthermore, he is able to measure the distance he has walked. However, he does not know exactly where in the forest he is. What is the best way to get out of the forest, put off the emergency call, and then get back to the injured hiker?"