The Lagrangian capacity of toric domains

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Goal

Basics

Results
Conjecture ([Per22, Conjecture 6.24])

If $X_\Omega$ is a convex or concave toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Goal

To motivate and prove the conjecture (in some special cases).
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Goal

Basics

Results
Definition 2.1

1. The **moment map** is the map $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n_{\geq 0}$ given by $\mu(z_1, \ldots, z_n) := \pi(|z_1|^2, \ldots, |z_n|^2)$.

2. A **toric domain** is a star-shaped domain $X$ of the form $X = X_\Omega := \mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^n_{\geq 0}$.

3. The **diagonal** of $X_\Omega$ is $\delta_\Omega := \sup\{a \mid (a, \ldots, a) \in \Omega\}$.

Example 2.2

\[ P(a) := \{z \in \mathbb{C}^n \mid \forall j = 1, \ldots, n: \pi|z_j|^2 \leq a\} \quad \text{(cube)} \]
\[ N(a) := \{z \in \mathbb{C}^n \mid \exists j = 1, \ldots, n: \pi|z_j|^2 \leq a\} \]
\[ \quad \text{(nondisjoint union of cylinders)} \]
Definition 2.3 ([CM18, Section 1.2])
Let \((X, \omega)\) be a symplectic manifold. If \(L\) is a Lagrangian submanifold of \(X\), then we define the **minimal symplectic area of** \(L\) by

\[
A_{\text{min}}(L) := \inf \{ \omega(\sigma) \mid \sigma \in \pi_2(X, L), \, \omega(\sigma) > 0 \}.
\]

Definition 2.4 ([CM18, Section 1.2])
The **Lagrangian capacity** of \((X, \omega)\) is

\[
c_L(X) := \sup \{ A_{\text{min}}(L) \mid L \subset X \text{ is an embedded Lagrangian torus} \}.
\]

Definition 2.5
The **cube capacity** is given by

\[
c_P(X, \omega) := \sup \{ a \mid \exists \text{ symplectic embedding } P^{2n}(a) \to X \}.
\]
Lemma 2.6
If $X$ is a star-shaped domain, then $c_L(X) \geq c_P(X)$.

Proof.
Let $\iota: P(a) \longrightarrow X$ be a symplectic embedding, for some $a > 0$. We want to show that $c_L(X) \geq a$. Define $T := \mu^{-1}(a, \ldots, a) \subset \partial P(a)$ and $L := \iota(T) \subset X$. Then, $c_L(X) \geq A_{\min}(L) = A_{\min}(T) = a$. \qed

Figure: Proof of $c_L(X) \geq c_P(X)$ for $X = X_\Omega$
Lemma 2.7

*If* $X_\Omega$ *is a convex or concave toric domain, then* $c_P(X_\Omega) \geq \delta_\Omega$.

**Proof.**

Since $X_\Omega$ is convex or concave, we have $P(\delta_\Omega) \subset X_\Omega \subset N(\delta_\Omega)$. The result follows since $c_P(X_\Omega) := \sup\{a \mid \exists P(a) \hookrightarrow X_\Omega\}$.

**Figure:** If $X_\Omega$ is convex or concave then $P(\delta_\Omega) \subset X_\Omega \subset N(\delta_\Omega)$
We now consider the results by Cieliebak–Mohnke for the Lagrangian capacity of the ball and the cylinder.

Proposition 2.8 ([CM18, Corollary 1.3])

The Lagrangian capacity of the ball is

\[ c_L(B^{2n}(1)) = \frac{1}{n} = \delta_\Omega(B^{2n}(1)). \]

Proposition 2.9 ([CM18, p. 215-216])

The Lagrangian capacity of the cylinder is

\[ c_L(Z^{2n}(1)) = 1 = \delta_\Omega(Z^{2n}(1)). \]
Conclusion

$X_\Omega$ is a convex or concave toric domain $\implies c_L(X_\Omega) \geq \delta_\Omega$

$X_\Omega$ is the ball or the cylinder $\implies c_L(X_\Omega) = \delta_\Omega$

Conjecture 2.10 ([Per22, Conjecture 6.24])

*If $X_\Omega$ is a convex or concave toric domain then*

$$c_L(X_\Omega) = \delta_\Omega.$$
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To prove our results about the Conjecture 2.10, we will need to use the following symplectic capacities.

McDuff–Siegel capacities \( \tilde{g}_k^{\leq \ell} \) [MS22]

Higher symplectic capacities \( g_k^{\leq \ell} \) [Sie20]

Gutt–Hutchings capacities \( c_k^{GH} \) [GH18]

for \( k, \ell \in \mathbb{Z}_{\geq 1} \). We will only need to consider these capacities for \( \ell = 1 \), i.e. \( \tilde{g}_k^{\leq 1}, g_k^{\leq 1} \).
Theorem 3.1 ([Per22, Theorem 6.40])

If $(X, \lambda)$ is a Liouville domain and $k \geq 1$ then $c_L(X) \leq \tilde{g}^{\leq 1}_k(X)/k$.

Proof sketch.

1. By definition of $c_L$, it suffices to assume that $L \subset X$ is an embedded Lagrangian torus and to prove that there exists a disk $D$ with boundary on $L$ with “small” symplectic area.

2. By definition of $\tilde{g}^{\leq 1}_k$, there exists a sequence $u_t$ of $J_t$-holomorphic curves with bounded energy and satisfying a tangency constraint.

3. By the SFT compactness theorem, $u_t$ converges to a broken holomorphic curve $F = (F_1, \ldots, F_N)$ (neck stretching along $S^*L$).

4. One of the components of the broken holomorphic curve $F$ will be the desired disk.
Theorem 3.2 ([Per22, Theorem 6.41])

If $X_\Omega$ is a 4-dimensional convex toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Proof.

For every $k \in \mathbb{Z}_{\geq 1}$,

\[
\begin{align*}
\delta_\Omega & \leq c_P(X_\Omega) \quad \text{[by Lemma 2.7]} \\
& \leq c_L(X_\Omega) \quad \text{[by Lemma 2.6]} \\
& \leq \tilde{g}_{k}^{\leq 1}(X_\Omega)/k \quad \text{[by Theorem 3.1]} \\
& = c_{k}^{GH}(X_\Omega)/k \quad \text{[dim 4 and [MS22, Proposition 5.6.1]]} \\
& \leq c_{k}^{GH}(N(\delta_\Omega))/k \quad \text{[}$X_\Omega$ is convex, hence $X_\Omega \subset N(\delta_\Omega)$]}
\end{align*}
\]

\[
= \delta_\Omega(k + 1)/k \quad \text{[by [GH18, Lemma 1.19]].}
\]
Theorem 3.3 ([Per22, Theorem 7.64])

If $X$ is a Liouville domain such that $\pi_1(X) = 0$ and $2c_1(TX) = 0$ then $g^\leq_1(X) = c^\text{GH}_k(X)$.

Proof sketch.

1. Let $E = E(a_1, \ldots, a_n)$ be a “skinny” ellipsoid such that there exists a strict exact symplectic embedding $\phi: E \to X$.

2. By definition of $c^\text{GH}_k$ and $g^\leq_1$ (and the Bourgeois–Oancea isomorphism), it suffices to show that $\#^\text{vir} \mathcal{M}^J_E(\gamma)\langle T^{(k)}x \rangle \neq 0$.

3. Show that $\mathcal{M}^J_E(\gamma)\langle T^{(k)}x \rangle$ is transversely cut out. This implies that $\#^\text{vir} \mathcal{M}^J_E(\gamma)\langle T^{(k)}x \rangle = \# \mathcal{M}^J_E(\gamma)\langle T^{(k)}x \rangle$.

4. Compute explicitly that $\# \mathcal{M}^J_E(\gamma)\langle T^{(k)}x \rangle \neq 0$ (curves in this moduli space are polynomials).
Theorem 3.4 ([Per22, Theorem 7.65])

Assume that a suitable virtual perturbation scheme exists. If \( X_\Omega \) is a convex or concave toric domain then \( c_L(X_\Omega) = \delta_\Omega \).

Proof.

\[
\begin{align*}
\delta_\Omega &\leq c_P(X_\Omega) & \quad \text{[by Lemma 2.7]} \\
&\leq c_L(X_\Omega) & \quad \text{[by Lemma 2.6]} \\
&\leq \tilde{g}^\leq_1(X_\Omega)/k & \quad \text{[by Theorem 3.1]} \\
&\leq g^\leq_1(X_\Omega)/k & \quad \text{[by [MS22, Section 3.4]]} \\
&= c_k^{GH}(X_\Omega)/k & \quad \text{[by Theorem 3.3]} \\
&\leq c_k^{GH}(N(\delta_\Omega))/k & \quad \text{[\( X_\Omega \) is convex, hence \( X_\Omega \subset N(\delta_\Omega) \)]} \\
&= \delta_\Omega(k + n - 1)/k & \quad \text{[by [GH18, Lemma 1.19]]}.
\end{align*}
\]
\[ c_L(X) \leq \inf_k \frac{\tilde{g}_{k}^{\leq 1}(X)}{k} \]

\[ g_{k}^{\leq 1}(X) = c_k^{GH}(X) \]

\[ c_L(X_{\Omega}) = \delta_{\Omega} \]

Thank you for listening!
References


