Integer-Valued Gromov–Witten Type Invariants

Guangbo Xu
(joint with Shaoyun Bai)

Texas A&M University

Symplectic Zoominar
June 3, 2022
1. Gromov–Witten invariants and virtual fundamental cycles
2. Normally polynomial sections of Fukaya–Ono
3. The integral Euler cycle and the integral invariants
4. The transversality condition
I. Gromov–Witten invariants and virtual fundamental cycles
Gromov–Witten invariants

- Fix a compact symplectic manifold \((X, \omega)\) and a compatible almost complex structure \(J\). Fix \(A \in H_2(X; \mathbb{Z})\) and \(g, n \geq 0\).
- Consider the moduli space of \(J\)-holomorphic maps

\[
\overline{M}_{g,n}(X, J; A).
\]

- The Gromov–Witten invariants are defined via the virtual fundamental class

\[
[\overline{M}_{g,n}(X, J, A)]^{\text{vir}} \in H_*(X^n \times \overline{M}_{g,n}; \mathbb{Q}).
\]
We need to additional structures to construct the virtual fundamental class.

**Definition (Fukaya–Ono)**
A Kuranishi chart on $\overline{\mathcal{M}}$ is $(G, U, W, S, \psi)$ where $G$ is a finite group, $U$ is an (open) $G$-manifold, $W$ is a $G$-representation, $S : U \to W$ is a $G$-equivariant map, and $\psi : S^{-1}(0)/G \to \overline{\mathcal{M}}$ is a homeomorphism onto an open set.

![Diagram](attachment:image.png)
One can cover $\mathcal{M}$ by such Kuranishi charts $(G_\alpha, U_\alpha, W_\alpha, S_\alpha)$ with certain coordinate changes. The union of a nice collection of charts gives a virtual neighborhood of $\mathcal{M}$

$$\mathcal{M} = \bigcup_{\alpha} S^{-1}_\alpha(0)/G_\alpha \subset \bigcup_{\alpha} U_\alpha/G_\alpha.$$ 

If all $S_\alpha : U_\alpha \to W_\alpha$ are transverse to $0 \in W_\alpha$, then $\mathcal{M}$ is an orbifold and hence carries an a priori rational fundamental class of degree $\dim U_\alpha - \text{rank} W_\alpha$.

If $S_\alpha$ is not transverse then we hope to slightly perturb it to achieve transversality.

In general there are obstructions to achieve equivariant transversality by perturbation. One then uses multivalued perturbations to achieve transversality and define a rational fundamental class.

II. Normally polynomial perturbations of Fukaya–Ono
• Consider a chart \((G, U, W, S)\). Let \(U^G \subseteq U\) be the \(G\)-fixed point set.

• Assume the normal bundle to \(U^G\) is trivial: \(NU^G \cong U^G \times V\) where \(V\) is a \(G\)-representation containing no trivial \(G\)-summand.

• Near \(U^G\) the map \(S\) corresponds to a family of maps from \(V\) to \(W\)

\[
f : U^G \rightarrow C^\infty(V, W) \bigg|_G,
\]

\[
f_x = S|_{\{x\} \times V}.
\]
Equivariant transversality fails

- Decompose the obstruction space

\[ W = W_{\text{trivial}} \oplus W_{\text{nontrivial}} = \hat{W} \oplus \check{W}, \quad S = (\hat{S}, \check{S}) \]

where \( G \) acts trivially on \( \hat{W} \) and \( \check{W} \) has no trivial summand.

- We can do single-valued perturbation to make \( \hat{S} \) transverse. Hence we can replace the chart by

\[ (G, \hat{S}^{-1}(0), \hat{W}, \hat{S}|_{\hat{S}^{-1}(0)}). \]

Hence we can assume \( W \) has no trivial summand.

- Then any equivariant map \( f \in C^\infty(\mathcal{V}, \mathcal{W})^G \) must vanish at \( 0 \in \mathcal{V} \). So for all single-valued perturbation \( S_f : U \to \mathcal{W}, U^G \subset S_f^{-1}(0) \).

- However the dimension of \( U^G \) might be larger than the virtual dimension.
Fukaya–Ono: If both $V$ and $W$ are complex representations, then consider single-valued perturbations $S_f$ corresponding to

$$f : U^G \to \operatorname{Poly}_d^G(V, W) := \left\{ \text{Equivariant polynomial maps } P : V \to W \text{ of degrees } \leq d \right\}.$$
Lemma (Fukaya–Ono, 1997)

For $d$ sufficiently large, for a generic normally polynomial perturbation $S_f$

1. The free part $(S_f^{-1}(0))_{\text{free}} := S_f^{-1}(0) \cap (NU^G)_{\text{free}}$ is transverse.

2. The boundary of the free part

$$(S_f^{-1}(0))_{\text{free}} \setminus (S_f^{-1}(0))_{\text{free}}$$

is contained in the union of submanifolds of codimension two or higher.
Conjecture (Fukaya–Ono, 1997)

The moduli space of stable maps has a kind of “normally complex” Kuranishi structure. For a kind of “normally polynomial perturbation” one can define integer-valued Gromov–Witten invariants which morally counts curves with trivial automorphism groups.

Using similar type of perturbations one can define Hamiltonian Floer homology in integer coefficients and prove Arnold conjecture over $\mathbb{Z}$. 
III. The integral Euler cycle and integral invariants
• Let $\mathcal{U}$ be an effective orbifold and $\mathcal{W} \to \mathcal{U}$ is an orbifold vector bundle. Let $S : \mathcal{U} \to \mathcal{W}$ be a section with $S^{-1}(0)$ compact. We call $(\mathcal{U}, \mathcal{W}, S)$ a derived orbifold chart.
• Notice that locally around each $p \in S^{-1}(0)$ one can find a Kuranishi chart $(G_p, U_p, W_p, S_p)$.
• When $\mathcal{W}$ is (relatively) oriented, there is a well-defined Euler class

$$\chi^\text{orb}(\mathcal{U}, \mathcal{W}, S) \in H_*(\mathcal{U}; \mathbb{Q})$$

represented the zero locus of a small transverse multivalued perturbation of $S$.
• What Fukaya–Ono suggested is that when certain “normal complex structure” exists, there is a different class

$$\chi^\mathbb{Z}(\mathcal{U}, \mathcal{W}, S) \in H_*(\mathcal{U}; \mathbb{Z})$$

represented by a pseudocycle contained in the manifold part $\mathcal{U}_\text{mfd} \subset \mathcal{U}$. 
For simplicity we assume both $TU$ and $W$ are complex.

Any section $S : U \to W$ locally is an equivariant map

$$S_p : U_p \to W_p \cong \hat{W}_p \oplus \check{W}_p.$$ 

We consider those normally polynomial ones.

For each (small) chart $(G_p, U_p, W_p, S_p)$ centered at $p$, we want to perturb $S_p = (\hat{S}_p, \check{S}_p)$ to one for which $\hat{S}_p$ is induced by a generic family

$$f_p : U^G_p \to \text{Poly}^G_d(V_p, \check{W}_p).$$

We need a notion of transversality for such global perturbation so that locally it behaves as Fukaya–Ono expected.
Following Fukaya–Ono and inspired by a subsequent work by Brett Parker, we defined a notion of transversality for such sections.

**Theorem (Bai–X., 2022)**

There is a notion of strong transversality for normally polynomial perturbations satisfying the following conditions.

1. A generic normally polynomial perturbation is strongly transverse (we call them FOP perturbations).
2. For any FOP perturbation $S' : U \to W$,

   $$(S')^{-1}(0) \cap \mathcal{U}_{\text{mfd}} \subset U$$

   is a pseudocycle whose homology class is a (cobordism) invariant of $(U, W, S)$, denoted by

   $$\chi^{\text{FOP}}(U, W, S) \in H_*(U; \mathbb{Z}).$$
**Theorem (Abouzaid–McLean–Smith, 2021)**

For a compact symplectic manifold $X$, $n \geq 0$, $A \in H_2(X; \mathbb{Z})$, there is a complex derived orbifold chart $(\mathcal{U}, \mathcal{W}, S)$ (a global Kuranishi chart), which is canonical up to certain equivalence relations, such that

$$\overline{\mathcal{M}}_{0,n}(X, J; A) = S^{-1}(0).$$

**Theorem (Abouzaid–McLean–Smith, 2021)**

If $P \to S^2$ is a Hamiltonian fiber bundle with fibre being a compact symplectic manifold $(X, \omega)$, then there holds

$$H_*(P; \mathbb{Z}) \cong H_*(S^2; \mathbb{Z}) \otimes H_*(X; \mathbb{Z}).$$
**Integral invariants**

**Corollary (Bai–X., 2022)**

Recall the map

$$\text{ev} \times \text{st} : \overline{M}_{0,n}(X, J; A) \to X^n \times \overline{M}_{0,n}.$$  

The homology class

$$[\overline{M}_{0,n}(X, J, A)]_{\text{FOP}} := (\text{ev} \times \text{st})_* (\chi_{\text{FOP}}(U, W, S)) \in H_*(X^n \times \overline{M}_{0,n}; \mathbb{Z})$$

is a symplectic deformation invariant, i.e., independent of almost complex structure, the global chart, and the perturbation. Therefore, its pairing with integral cohomology classes defines an integer-version of Gromov–Witten invariants

$$\Psi_{g, n, A} : H^*(\overline{M}_{0,n}; \mathbb{Z}) \otimes H^*(X; \mathbb{Z}) \otimes \cdots \otimes H^*(X; \mathbb{Z}) \to \mathbb{Z}.$$  

It should match the usual genus zero Gromov–Witten invariants for semi-positive $X$. 
Theorem (Bai–X., 2022)

For any finite group $\Gamma$, let $U_\Gamma \subset U$ be the set of points whose isotropy group is $\Gamma$. Then for any FOP perturbation $S': U \to W$,

$$(S')^{-1}(0) \cap U_\Gamma$$

is a pseudocycle. Hence there is a symplectic deformation invariant

$$[\mathcal{M}_{0,n}(X, J, A)]_{FOP}^{\Gamma} \in H_{*}(X^n \times \overline{\mathcal{M}}_{0,n}; \mathbb{Z}).$$

Conjecture (Bai–X.)

The FOP cycles can be constructed for all genera; for each finite group $\Gamma$, $[\overline{\mathcal{M}}_{g,n}(X, J, A)]_{FOP}^{\Gamma}$ is morally supported in $\overline{\mathcal{M}}_{g,n}(X, J, A)_\Gamma$, the locus of stable maps of automorphism group $\Gamma$. Moreover, there exists a certain virtual sheaf $(V_\Gamma \to W_\Gamma)$ and an integral characteristic class $e(V_\Gamma \to W_\Gamma)$ such that

$$[\overline{\mathcal{M}}_{g,n}(X, J, A)]^{\text{vir}} = \sum_{\Gamma} \frac{1}{|\Gamma|} \left\{ [\overline{\mathcal{M}}_{g,n}(X, J, A)]_{FOP}^{\Gamma} \cap e(V_\Gamma \to W_\Gamma) \right\}.$$
• We reproved Abouzaid–McLean–Smith’s result without using generalized cohomology theories.

• The Gromov–Witten axioms for genus zero (such as WDVV equation/quantum cohomology) can be verified for this new invariant.

• It is straightforward to define the quantum Steenrod operation on general compact symplectic manifolds, extending the monotone case defined by Seidel and Wilkins.

• Our construction needs smooth Kuranishi structures. So far we do not have a sufficiently simple way to establish smoothness for higher genus case.

• If we can construct smooth Kuranishi structures on all moduli spaces of Floer trajectories, then one can define an integer-version of Hamiltonian Floer homology and prove the Arnold conjecture over $\mathbb{Z}$. 
IV. The transversality condition
Revisiting the local picture

- For a Kuranishi chart $G \xrightarrow{U} S \xrightarrow{W} G$. Assume $W$ has no trivial sum-mand.
- Recall the universal zero set is an affine variety

$$Z := Z_d^G(V, W) := \{(v, P) \in V \times \text{Poly}^G_d(V, W) \mid P(v) = 0\}.$$  

Denote

$$Z_{\text{free}} := Z \cap (V_{\text{free}} \times \text{Poly}^G_d(V, W)).$$

- Because $Z_{\text{free}} \subset Z$ is a Zariski open set, the boundary

$$\overline{Z_{\text{free}}} \setminus Z_{\text{free}}$$

is the union of smooth algebraic sets of codimension two or higher.
- We need maps to be transverse to both $Z_{\text{free}}$ and boundary strata.
**Theorem (Thom, Whitney, Łojasiewicz)**

Given a smooth complex algebraic variety $M$ and an algebraic subset $Z \subset M$, there exists a “canonical” Whitney stratification, i.e., a decomposition

$$Z = \bigsqcup_{\alpha} Z_\alpha$$

into strata, which satisfy “Whitney condition (b)”. Moreover, each $Z_\alpha$ is a smooth algebraic set.

**Lemma (Bai–X.)**

There exists a canonical Whitney stratification on $Z \subset V \times \text{Poly}_d^G(V, W)$, such that each stratum is contained $V_H \times \text{Poly}_d^G(V, W)$ where

$$V_H = \{v \in V \mid G_v = H\}, \quad H < G.$$
A family of polynomial maps $f : U^G \to \text{Poly}_G^G(V, W)$ is strongly transverse if the associated map $F : V \times U^G \to V \times \text{Poly}_G^G(V, W)$ is transverse to each stratum of $Z$.

Then $f$ is strongly transverse implies that

$$(S^{-1}_f(0))_{\text{free}} \setminus (S^{-1}_f(0))_{\text{free}} \subseteq \bigcup \text{manifolds of codimension } \geq 2.$$

We chose the Whitney stratification in such a careful way in order to settle the following two subtle issues.

1. Does strong transversality for $d$ imply strong transversality for $d' > d$?
2. Does strong transversality for $G$ imply strong transversality for $H < G$?
• When considering Fukaya–Ono’s proposal, Brett Parker (2013) introduced a class of “nice” Whitney stratifications on $Z$ and found a method of relating nice Whitney stratifications on $Z$ for different $d$ and $G$.

• However Parker did not provide a coherent way to choose nice Whitney stratifications nor provide enough details for constructing global transverse perturbations with his version of Kuranishi structure.

• By modifying Parker’s method, we proved that our strong transversality condition is intrinsic, i.e., being independent of large $d$ and being an open condition.
THANKS!