Locality & deformations in relative symplectic cohomology

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Basic Setup

(M,ω) - symp mfld, closed or geom. bdd, $c_1(M)=0$

$K \subseteq M$ compact.

$\rightsquigarrow$ $SH^*_M(K)$ - SH of K rel M

Structures

- $\mathbb{Z}$-graded module over Novikov ring

$\Lambda_{nov} := \{ \sum_i a_i T^{\lambda_i} \mid a_i \in \mathbb{R},\ \lambda_i \in \mathbb{R}_+,\ \lambda_i \to \infty \}$

- Carries non-Archimedean norm $|\cdot|$, complete w.r.t. $|\cdot|$

- BV algebra str.

- $K_1 \subseteq K_2 \rightsquigarrow SH^*_M(K_2) \longrightarrow SH^*_M(K_1)$

  Functorial BV-algebra homomorphism
  norm non-increasing.

Formal definition

Consider $\mathcal{H}_K = \{ H \in C^\infty(S^1 \times M) \mid H_{S^1 \times K} < 0 \}$

In favorable cases:

$$SH^*_M(K) = \varprojlim_{\lambda} \varinjlim_{H \in \mathcal{H}_K} HF^*_\lambda(H)$$

λ-truncated homology

- the maps in $\mathcal{H}_K$ are monotone homotopies

$s \longmapsto H_s$, $\partial_s H_s \geq 0$

(In general, need a derived inverse limit)

$$SH^*_M(K) = HF^*(\mathcal{X}_{M \setminus K}) \leftarrow$$

∞ on $M \setminus K$

0 on K.

Q: Is relative SH computable?

In general, hopeless

1) We can't solve Hamilton's equation

2) We can't solve Floer's equation

the situation improves dramatically for <u>integrable systems</u>

<u>Consider:</u>

$\pi : M \twoheadrightarrow B$ proper surjection

M: dim $2n$

B: smooth n-dim mfld

s.t.

$$\{f_1 \circ \pi, f_2 \circ \pi\} = 0, \ \forall f_1, f_2 \in C^\infty(B, \mathbb{R})$$

i.e. $\pi^{-1}(b)$ - Lag torus, for generic $b \in B$

<u>Aim:</u> $P \subseteq B$ - cpt subset

Study $SH^*_M(\pi^{-1}(p))$

Motivation

Varolgunes ('20): Mayer Vietoris property relating $QH^*(M)$ & $SH_M(\pi^{-1}(P_i))$ for $\{P_i\}$ a cover of B

Note: only works for "commutative covers". Not, e.g., for cover by balls.

Basic case:

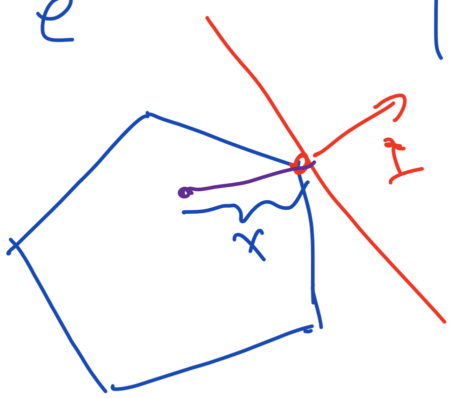
$M = T^*T^n = \underset{p}{\mathbb{R}^n} \times \underset{q}{T^n}$, $B = \mathbb{R}^n$, $\pi(p,q) = p$

$P \subseteq B$ - convex polytope with rational slopes

Over the Novikov field: (over T)

$$SH^*_M(\pi^{-1}(P)) = \Lambda_{nov}\Big[\underbrace{z_1^{\pm 1}, \dots, z_n^{\pm 1}}_{\deg 0}, \underbrace{\partial z_1, \dots, \partial z_n}_{\deg 1}\Big] \Big/ (\partial z_i)^2 = 0$$

$$|z^I| = e^{\langle I, x\rangle} \qquad |z_i \partial z_i| = 1$$



Formal Laurent-series converging on

$$\underbrace{\{\, x \in (\Lambda^*_{nov})^n \mid val(x) \in P \,\}}_{\text{mirror dual to } \pi^{-1}(P)} \subseteq (\Lambda^*_{nov})^n$$

BV operator: dual to d under iso. defined by $\Omega = \dfrac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}$

The general case:

M closed or geom. bdd,

π may have singular fibers

focus on $P \subseteq B_{reg} \subseteq B$

integral convex polytope. no singular values

$$
\begin{tikzcd}
 & M \\
\pi^{-1}(P) \arrow[ur, hook] \arrow[dr, hook] & \\
 & T^*T^n
\end{tikzcd}
$$

Q. Can we relate $SH^*_M(\pi^{-1}(P))$ & $SH^*_{T^*T^n}(\pi^{-1}(P))$?

2 locality results

1) (G., Varolgunes)

Locality for complete embeddings

2) (G., - In preparation)

A locality spectral sequence.

Locality for complete embeddings

Def

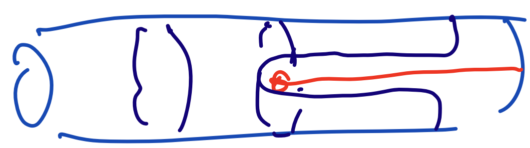
A <u>complete embedding</u> is a symplectic embedding $i: \mathcal{Y} \longrightarrow \mathcal{X}$ where $\mathcal{X}, \mathcal{Y}$ - geom bdd.

i.e. carry a <u>complete</u> compatible metric g s.t.

$$|\sec_g| < C < \infty$$
$$|\mathrm{inj}_g| > c > 0$$

<u>examples</u>

1) $\mathbb{R}\times S^1 \setminus [0,\infty)\times\{0\}$



2) $T^*M \hookrightarrow T^*N$ for $M \hookrightarrow N$ an open inclusion of sm. mflds.

Non-examples:

1) $B_1(0) \subseteq \mathbb{C}$

2) $\mathbb{C}^* \hookrightarrow \mathbb{C}$

3) $\mathbb{C}^*\times\mathbb{C} \hookrightarrow \mathbb{C}^2$ (non-trivial!)

consider

$$\begin{tikzcd} & & \mathcal{X} \\ K \arrow[urr, hook] \arrow[drr, hook] & & \\ & & \mathcal{Y} \arrow[uu, "i"] \end{tikzcd}$$

cpt $\in K$

$\longleftarrow$ Complete embedding

Thm (G- Varolgunes)

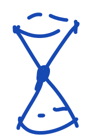
There is a canonical iso.

$$i_* : SH^*_{\mathcal{Y}}(K) \longrightarrow SH^*_X(i(K)).$$

Example: symplectic cluster manifolds

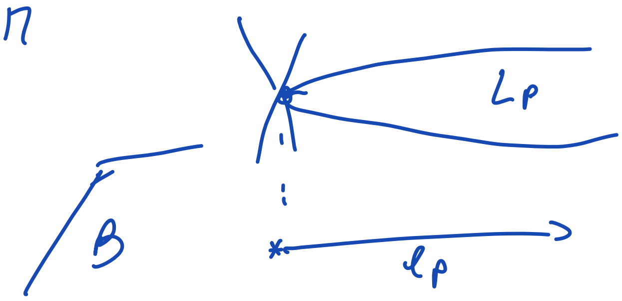
M - 4-dim, $B \simeq \mathbb{R}^2$, $\pi : M \longrightarrow B$

$\# \operatorname{Crit} \pi < \infty$, all singularities are nodal

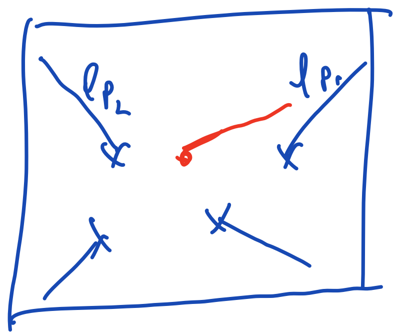


+ there's a choice of pairwise disjoint properly embedded Lagrangian tails

L_p, $p \in \operatorname{Crit} \pi$



s.t

$B =$ 

integral affine rays can be continued at least until they hit a ray

Theorem (G., Varolgunes)

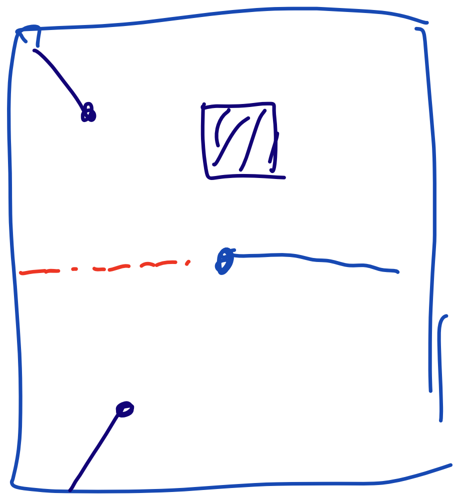
$M \setminus \bigcup_p L_p \simeq T^* T^2$.

The data $(\pi, \{L_p\})$ is called a **cluster presentation** for M.

Note:

M has many different cluster presentations.

$\Rightarrow$ many different complete embeddings containing a given K



$$\rightsquigarrow \quad \begin{tikzcd} SH^*_M(K) \arrow[r, bend left, "i^1_*"] \arrow[r, bend right, "i^2_*"'] & SH^*_{T^*T^2}(K) \end{tikzcd}$$

In progress (G-, Varolgunes): $i^1_* \circ (i^2_*)^{-1}$ is a cluster transformation.

<u>Limitations:</u> 1) If M is closed, there are no complete embeddings

2) We need more tools to study nbhds of singular fibers.

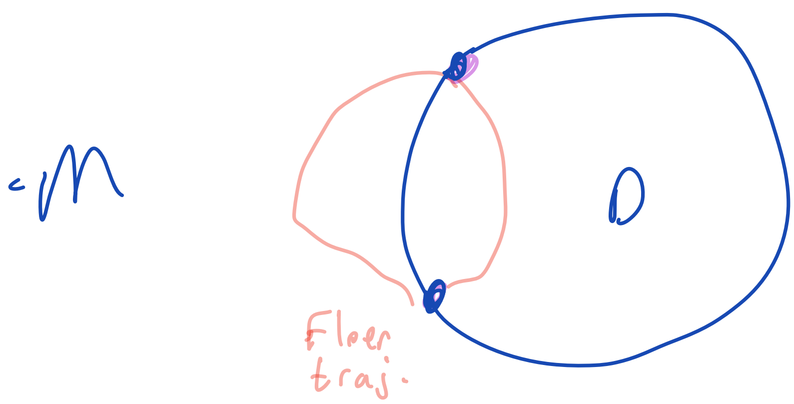
$SH^*_M(D)$ as a deformation of $SH^*_D(D)$

Idea

Let (D, θ) - Liouville domain,

$i : (D, \theta) \hookrightarrow M$, $i^* \omega = d\theta$

$\rightsquigarrow [\omega, \theta] \in H^2(M, i(D))$



Integrated max. principle:

$\langle [\omega, \theta], u \rangle > 0$ for trajectories u

Connecting Reeb orbits in D & going outside.

In fact: $\langle [u, \theta], u \rangle \geq \hbar$, for some $\hbar \in \mathbb{R}_+$ $\rightsquigarrow$ filter by $\langle [u,\theta], u \rangle$

$\rightsquigarrow$ spectral sequence weak convergence

$$E_1 = SH^*_{\bar{D}}(D; R) \otimes \Lambda_{nov} \Rightarrow SH^*_M(D)$$

The Locality spectral sequence (L.s.s.)

<u>Pblm:</u> L.s.s. doesn't necessarily converge

<u>Example:</u> $M = S^2$ of area 1

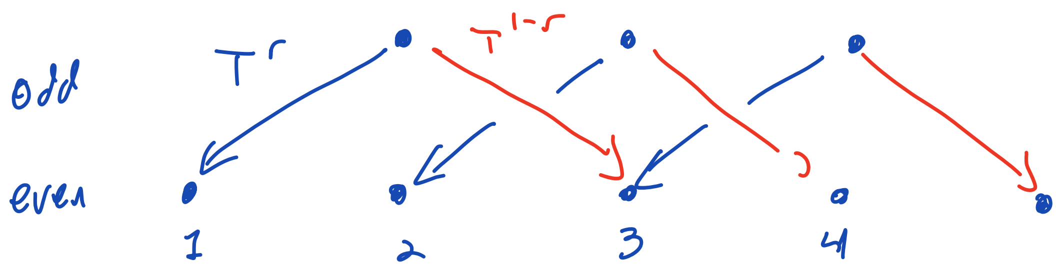
D = disc of area $r > \frac{1}{2}$

$E_1 = 0$, $SH^*_M(D) = QH(S^2) \neq 0$.

reason:

$d(x^i \partial_x) = T^r x^i + T^{1-r} x^{i+2}$

x^i, $x^{i-1}\partial_x$
even odd



$d = d_0 + \underbrace{d_{def}}_{e^{r-1}}$ $\qquad |d_{def}| > e^{-\beta(d_0)}$

boundary depth

Theorem (C - in preparation)

There is an $\hbar > 0$ s.t. if $\qquad r \in (0,1)$

$\beta(SC^*_{\bar{D}}(r \cdot D)) < \hbar$ and if the

L.S.S of $r \cdot D$ degenerates on

page 1, it converges.

(i.e. $SH^*_{\bar{D}}(r\cdot D) \simeq SH^*_{M}(r\cdot D)$)

Q: Can these hypotheses be verified?

Note i) for $P \subseteq \mathbb{R}^n$ integral affine convex $\beta(\pi^{-1}(P)) = 0$

2) Vanishing of β can be shown in many cases related to SYZ

$\Rightarrow$ suffices to verify degeneration on 1st page.

$F' \; SC^* = \; 1 \; SC^*$

Observe: The L.S.S involves a choice of real λ (to get a $\mathbb{Z}$-filtration)

define $\tau(D, \theta, m) = \inf\{\lambda \mid d_{1,\rho} \neq 0\}$ differential on E_1

$\in \mathbb{R} \cup \{\infty\}$

Note:

If $\tau = \infty$ then locality holds!

Some properties:

1) If $\beta(D) < \infty$

$$\tau(r \cdot D, \theta, m) = \tau(D, \theta, m) \qquad r \in (0,1)$$

2) $$\tau(D, \underbrace{\theta + df}_{\text{if Liouville}}, m) = \tau(D, \theta, m)$$

$\leadsto$ $\tau : V \subseteq H^1(D, \mathbb{R}) \longrightarrow \mathbb{R}$

nbhd of 0

3) τ is locally an infimum of linear functions: $\langle [W, \partial] \quad u \rangle$

Floer trajectories

$\Rightarrow \tau$ is concave !

4) if $D_1 \hookrightarrow D_2$ and

$SH^*_{\bar{b}}(D_2) \longrightarrow SH^*_{\bar{b}}(D_1)$ is injective

$\Rightarrow \tau_1 \leq \tau_2$.

Call an embedding as in 4) <u>SH-essential</u>

<u>Example:</u> Positive singularity in 3D

$$M = \mathbb{C}^3 \setminus \{z_1 z_2 z_3 = 1\}$$

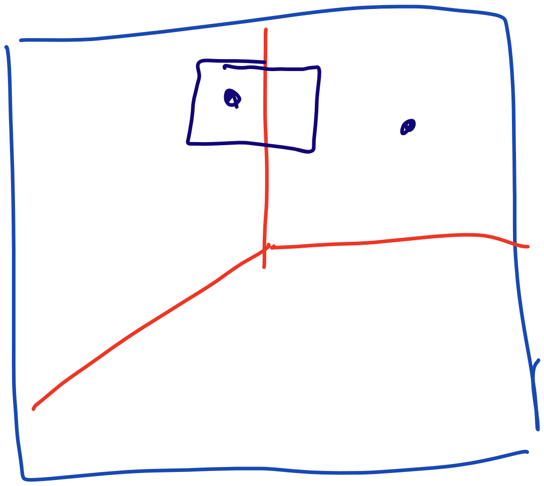
$$(\mathbb{C}^*)^2$$

$$\downarrow$$

$$\mathbb{R}^3$$

$$\pi(z_1, z_2, z_3) = \left(|z_1|^2 - |z_3|^2,\ |z_2|^2 - |z_3|^2,\ \log \|1 - z_1 z_2 z_3\|\right)$$

singular values



$x_3 = 0$

regular fibers are contained in a complete $T^* \mathbb{T}^3$ (for ω-completed)

$\Rightarrow$? (regular fiber) $= \infty$

nbhd of a generic singular point

$\simeq ([a,b] \times S^1) \times D_1$

nbhd of focus-focus singularity completely understood $\beta(D_1) = 0$, & fibers are SH essential

+ Künneth formula.

$\Rightarrow$ locality for generic singular fiber. Note: No complete embedding of $T^*S^1 \times D_1$!

Rmk:

Using an additional tool (homological perturbation algorithm w. in progress)

This can be used to show that for Q a nbhd of the vertex, P a nbhd of a regular fiber

$$SH^0(\pi^{-1}(Q)) \longrightarrow SH^0(\pi^{-1}(P))$$

is surjective up to inverses (i.e. is like restriction of analytic functions to a sub-domain)

Thank you!