Locality & deformations in relative symplectic cohomology

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Basic Setup

$(\mathfrak{m}, \omega)$-symplectic manifold, closed or geometrically bounded, $\mathfrak{c}_i (\mathfrak{m}) = 0$

$K \subseteq M$ compact

$\mathbb{W} \mapsto \mathcal{SH}^* (K) - \mathcal{SH}$ of $K$ rel $M$

Structures

- $\mathbb{Z}$-graded module over Novikov ring

$\Lambda_{\text{Nov}}^0 = \left\{ \sum_i a_i T^{\lambda_i} \mid a_i \in \mathbb{R}, \lambda_i \in \mathbb{R}_+, \lambda_i \to \infty \right\}$

- Carries non-Archimedean norm $\| \cdot \|$, complete w.r.t. $\| \cdot \|

- BV algebra str.

- $K_1 \subseteq K_2 \mapsto \mathcal{SH}^* (K_1) \to \mathcal{SH}^* (K_2)$

  Functorial BV-algebra homomorphism norm non-increasing.
**Formal definition**

Consider \( \mathcal{H}_k = \{ h \in C^\infty(S^1 \times M) \mid H_{S^1, k} < 0 \} \)

In favorable cases:

\[
\text{SH}^*_m(X) = \lim_{\lambda \to \text{truncational homology}} \lim_{\lambda \to \text{Homotopy}} \text{HF}_\lambda^*(H)
\]

the maps in \( \mathcal{H}_k \) are monotone homotopies

\( S(-) \sim H_5 \), \( 2^5 H_5 \Rightarrow \)

(In general, need a derived inverse limit)

\[
\text{SH}^*_m(X) = \text{HF}^*(X_{m \times k}) = \nu \circ \psi \circ \nu
\]

on \( m \times k \)

on \( k \)
Q: Is relative $\mathcal{SH}$ Computable?

In general, hopeless

1) We can't solve Hamilton's equation

2) We can't solve Floer's equation

the situation improves dramatically for integrable systems

Consider:

\[ \Pi : M \to B \text{ proper surjection} \]

\[ \dim M \geq \text{smooth } n - \dim M \text{fld} \]

s.t.

\[ df_1 \circ \Pi, f_2 \circ \Pi = 0, \forall f_1, f_2 \in C^0(B, \mathbb{R}) \]

i.e. $\Pi^{-1}(b)$ - Lag torus, for generic $b \in B$

Aim: $P \subseteq B$ - Cpt Subset
Study $\mathcal{S}H^*_m(\pi^{-1}(p))$

Motivation

Varolgunes (30): Mayer Vietoris Property relating $\mathcal{Q}H^*_m$

& $\mathcal{S}H^*_m(\pi^{-1}(p_i))$ for $p_i$ if a Cover of $B$

Note: only works for "commutative Covers" NOT, e.g., for cover by balls.
Basic case:

\[ M = \mathbb{T} \times \mathbb{T}^n = \mathbb{R}^n \setminus \mathbb{Q}^n \times \mathbb{T}^n \]
\[ B = \mathbb{R}^n, \quad \Pi(p, \theta) = p \]

\( p \in B \) - convex polytope with rational slopes

Over the Novikov field:

\[ S\text{Hom}^{\text{formal}}(\Pi^{-1}(p)) = \Lambda_{\text{nov}} \left[ \frac{\partial^{\pm 1} z_i}{\deg 0}, \frac{\partial^{\pm 1} z_i}{\deg 1}/(\partial z_i)^{\infty} \right] \]

\[ \prod I^I = e^{(I, X)} \quad \left( \prod z_i^\circ z_i \right) = 1 \]

Formal Laurent series converging on

\[ \left\{ x \in \left( \mathbb{A}^n_{\text{nov}} \right)^{\text{val}(x) \in p} \right\} \subseteq \left( \mathbb{A}^n_{\text{nov}} \right)^0 \]

mirror dual to \( \Pi^{-1}(p) \)
BV operator is dual to $d$ under iso, defined by $\mathcal{S} = \frac{dz_1 \ldots dz_n}{z_1 \ldots z_n}$

The general case:

$m$ closed or geo. $\text{bdd}$, $\Pi$ may have singular fibers

Focus on $p \in \text{Breg} \subseteq B$

integral convex no singular values

polytope

$\xrightarrow{\pi^{-1}(p)} M \xrightarrow{\pi} T^*T^m$

Q: Can we relate $\int \text{H}^*_m(\pi^{-1}(p))$ & $\int \text{H}^*_T(\pi^{-1}(p))$?
2 locality results

1) (a., Varolynnes)
   Locality for complete embeddings

2) (a., - In preparation)
   A locality spectral sequence
Locality for complete embeddings

Def

A complete embedding is a symplectic embedding \( i : y \to X \) where \( X, y \) - geom bdd.

i.e. carry a complete compatible metric \( g \) s.t. \[ | \sigma(g) | < C < \infty \]

\[ | \nabla g | > C > 0 \]

Examples

1) \( \mathbb{R} \times S^1 \setminus \{ \infty \} \times \partial S^1 \)

2) \( T^* M \to T^* N \) for \( M \to N \) an open inclusion of sm. mfd/s.

Non-examples:

1) \( B_1(0) \subseteq \mathbb{R} \)
2) \( C^* \subseteq C \)
3) \( C^* \times C \to \mathbb{C}^* \)
   (non-trivial !)
Consider \( \pi : X \to Y \)

**Thm (G. Varolynes)**

There is a canonical iso.

\[ i_* : \mathbb{H}^*_{\text{alg}}(X) \to \mathbb{H}^*_{\text{alg}}(i(X)) \]

**Example:** Symplectic cluster manifolds

\[ M = \text{hyp. dim.}, \quad B = \mathbb{R}^2, \quad \Pi : M \to B \]

\# Crit \( \Pi < \infty \), all singularities are nodal.

There's a choice of pairwise disjoint properly embedded Lagrangian tails \( L_p, \quad p \in \text{Crit} \Pi \)
Theorem (G. Varolgunes)

\[
\mathcal{M}(\mathbb{V}, \mathbb{L}, \mathbb{E}, \mathbb{T}) \subseteq \mathbb{T} \times \mathbb{T}. 
\]

The data \((\Pi, \mathcal{IL}_{\mathcal{E}})\) is called a cluster presentation for \(\mathcal{M}\).

Note:

\(\mathcal{M}\) has many different cluster presentations.

\(\mathcal{M}\) has many different complete embeddings containing a given \(k\).
In progress (A, Vao(genes) \cdot i_*^{\cdot 1} \cdot (i_*^{\cdot 2})^{-1} \\
is a cluster transformation.

Limitations:
2) If \( M \) is closed, there are no complete embeddings.
2) We need more tools to study

\( \text{n} \) \( \text{h} \) \( \text{o} \) \( \text{s} \) \( \text{i} \) \( \text{n} \) \( \text{g} \) \( \text{i} \) \( \text{l} \) \( \text{n} \) \( \text{a} \) \( \text{r} \) \( \text{o} \) \( \text{g} \) \( \text{e} \) \( \text{s} \) \( \text{e} \) \( \text{d} \)
$$\overline{SH^*_m(0)} \text{ as a deformation of } \overline{SH^*_D(0)}$$

**Idea**

Let $$(0, \theta)$$ - Liouville domain,

$$i : (0, \theta) \hookrightarrow M$$, $$i^*\omega = d\theta$$

$$\Rightarrow [w, \theta] \in H^2(M, i(0))$$

Integrated Max. principle:

$$\langle [w, \theta], u \rangle > 0 \text{ for trajectories } u$$
Connecting Reeb orbits in $O$ & going outside.

In fact: $\langle [U, o], U \rangle \cong$, for some $U \in \mathbb{R}_+$, $m$-filter by $\langle [U, o], U \rangle$

$m$-spectral sequence weak convergence

$$E_1 = \text{SH}^*(O; \mathbb{R}) \cong \text{SH}^*(O)$$

The Locality spectral sequence (L. s.s.)

Pblm: L. s.s. doesn't necessarily converge

Example: $m = S^2$ of area 1

$O = \text{disc of area } \approx \frac{1}{m}$

$$E_1 = 0, \text{ SH}^*_m (O) = \Omega H (S^2) \neq 0.$$
Reason:

\[ d(x^0, x) = T \mathcal{F} T^{-1} x^1 \]

Theorem (in preparation)

There is an \( \eta > 0 \) s.t. if

\[ \beta(\mathcal{C}(\mathcal{D}(\mathcal{D}(0))) \leq \eta \] and if the L.S.S. of \( \mathcal{D}(0) \) degenerates on
Page 2, it converges.

( i.e. \( SH^D_0 (r, D) \cong SH^D_m (r, D) \))

Q: Can these hypotheses be verified?

Note 1) for \( p \leq R^n \) integral affine convex \( B (\pi^{-1} (p)) = 0 \)

2) Vanishing of \( B \) can be shown in many cases related to SYZ

\( \Rightarrow \) suffices to verify degeneration on 1st page.
Observe: The L.S.S involves a choice of real \( \lambda \) (to get a \( \mathbb{R} \)-filtration)

\[ \mathcal{L}(D, \theta, \alpha) = \{ \lambda | \partial \mathcal{L}_D \neq 0 \} \]

\[ \in \mathbb{R} \cup \{ 0 \} \]

differential on \( E \)

Note:

If \( \exists = \infty \) then locality holds!

Some properties:

1) If \( \beta(D) < \infty \)

\[ \mathcal{L}(\gamma \cdot D, \theta, \alpha) = \mathcal{L}(D, \theta, \alpha) \cup \{ 0 \} \]

2) \( \mathcal{L}(D, \theta - \delta \mathcal{L}) = \mathcal{L}(D, \theta, \alpha) \)

if Liouville

\[ \text{Inf} : V \in H^1(D, \mathbb{R}) \rightarrow \mathbb{R} \]

nbd of 0

3) \( \mathcal{L} \) is locally an infimum of linear functions: \( \langle \mathcal{L}, \theta \rangle \in \mathbb{R} \)
Flare trajectories

\( \Rightarrow 7 \) is concave!

4) if \( D_1 \hookrightarrow D_2 \) and
\[ SH^x(D_2) \rightarrow SH^x(D_1) \text{ is injective} \]

\( \Rightarrow Z_1 < Z_2. \)

Call an embedding as a 4) \text{ SLT-essential}

\text{Example:} \text{ Positive singularity in 3D}

\[ m = \mathbb{C}^3 \setminus \{ z_1 z_2 z_3 = 1 \} \]

\[ \downarrow \]

\[ \mathbb{R}^3 \]

\[ \pi(z_1, z_2, z_3) = \left( 12z_1^2 - 12z_3^2, 12z_2^2 - 12z_3^2, \log \| z - z_1 z_2 z_3 \| \right) \]
Singular values

\[ x_3 = 0 \]

Regular fibers are contained in a complete \( T^* \mathbb{T}^3 \) (for \( W \)-complete)

\[ \Rightarrow \; \mathcal{O}(\text{regular fiber}) = \infty \]

\( \text{nbhd of a generic singular point} \)

\[ \cong \; ([a, b] \times S^1) \times 0. \]

\( \text{nbhd of focus-focus singularity} \)

Completely understood

\( \beta(0) = 0 \), \( \mathcal{B} \) fibers are \( SH \) essential
Kunneth formula.

=> locality for generic singular fiber. Note: no complete embedding of $T^{45} \times 0$.

Remark:

Using an additional tool (homological perturbation algorithm - in progress).

This can be used to show that for $a$ a nbhd of the vertex $p$ a nbhd of a regular fiber

$SH^0(\pi^{-1}(p)) \rightarrow SH^0(\pi^{-1}(p))$
is surjective up to inverses
(i.e. is like restriction of analytic functions to a sub-domain)

Thank you!