Contact Non-squeezing via Selective Symplectic Homology

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Contact geometry and size

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smooth topology $\subset$ contact geometry

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Fact

On a small scale, contact geometry does not remember the size.

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There is a non-trivial contact non-squeezing on a large scale.

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- Contact distribution on an Ustilovsky sphere is homotopic to the standard contact structure on the sphere if $p \equiv 1 \pmod{2 \cdot (2m)!}$.

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- Homotopy spheres admit Morse functions with precisely 2 critical points.
- Use gradient flow for smooth squeezing.

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The standard contact $\mathbb{R}^{4m+1}$ is not contactomorphic to any Ustilovsky sphere with a point removed.

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Theorem (Fauteux-Chapleau and Helfer)

There exist infinitely many pairwise non-contactomorphic tight contact structures on $\mathbb{R}^{2n+1}$ if $n > 1$.

Selective symplectic homology

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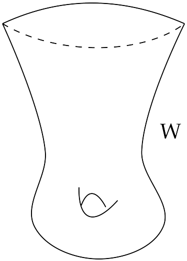
- W a Liouville domain



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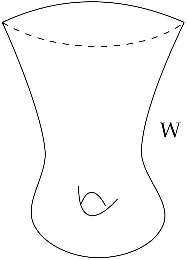

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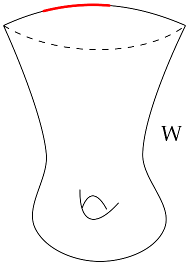

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Definition

A stop $\sigma : F \times \mathbb{C}_{Re<0} \to X$ on a Liouville manifold X is a proper codimension-0 embedding associated with a Liouville manifold F such that $\sigma^*\lambda_X = \lambda_F + \lambda_{\mathbb{C}} + df$, for a compactly supported f. Here, $\lambda_X, \lambda_F, \lambda_{\mathbb{C}}$ are the Liouville forms on X, F, and $\mathbb{C}_{Re<0}$, respectively.

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- Notions of a Liouville sector and of a stop on a Liouville manifold are essentially the same.
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- Every Liouville sector can be obtained by removing a stop from a Liouville manifold.

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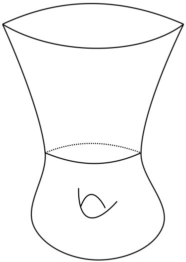

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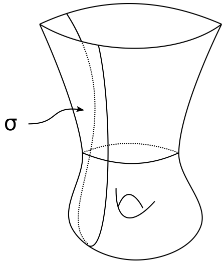

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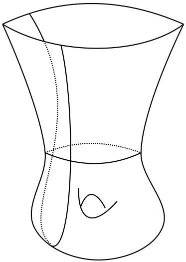

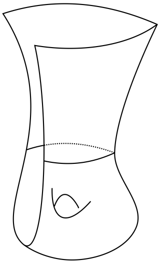

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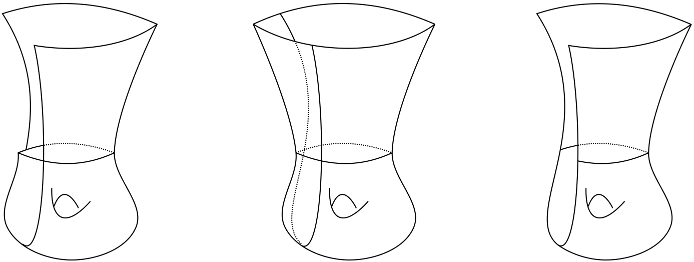

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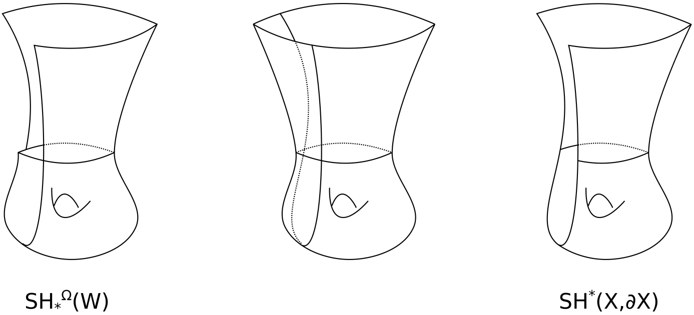

- $SH_*^\Omega(W)$ takes into account $W \cap \operatorname{im}\sigma$.

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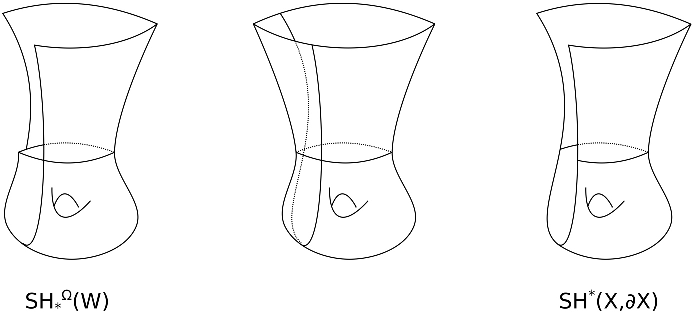

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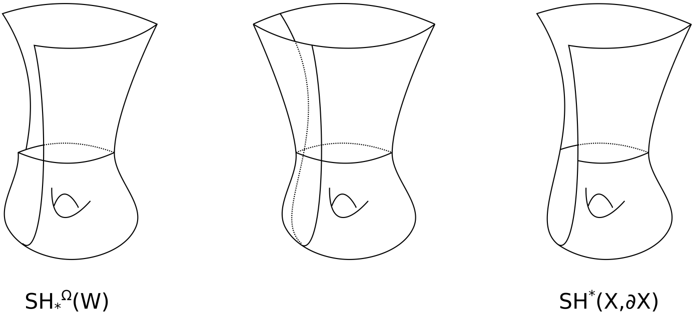

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- $SH_*^\Omega(W)$ is defined for any open subset $\Omega \subset \partial W$.

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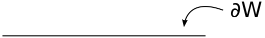

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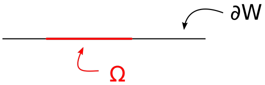

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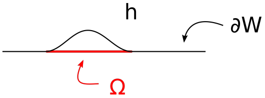

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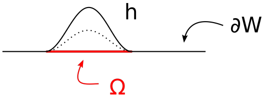

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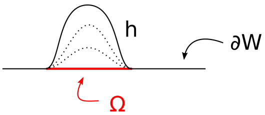

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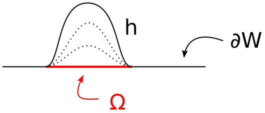

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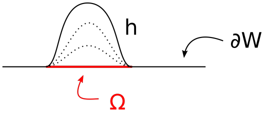

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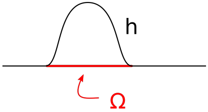

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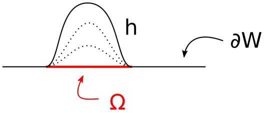

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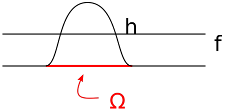

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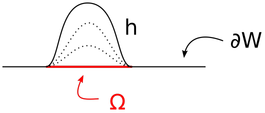

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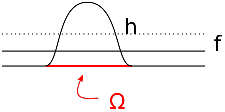

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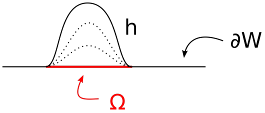

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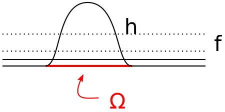

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$$SH_*^\Omega(W) \quad := \quad \varinjlim_{h \in \mathcal{H}_\Omega} \quad \varprojlim_{f \in \Pi(h)} \quad HF_*(h + f)$$

$HF_*(h)$ the Floer homology of a Hamiltonian $H : \hat{W} \to \mathbb{R}$ with slope h

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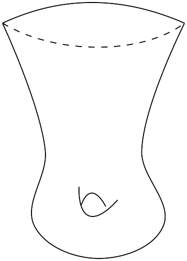

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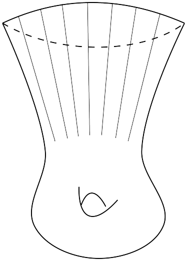

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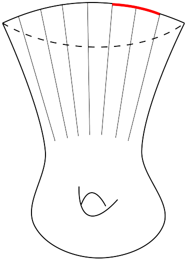

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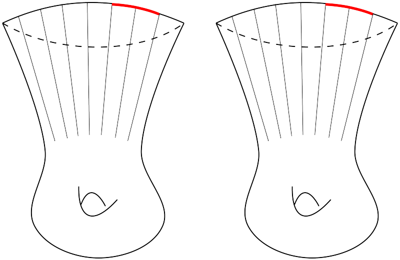

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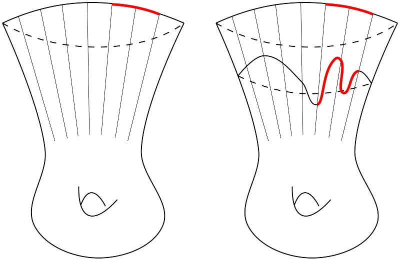

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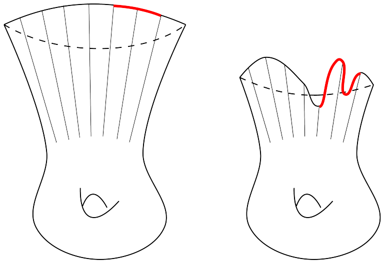

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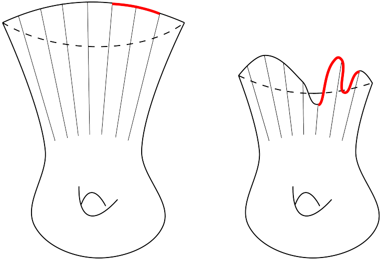

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Claim

In the situation above, there exists an isomophism

$$\mathcal{C}(\psi) : SH_*^{\Omega}(W) \to SH_*^{\varphi^{-1}(\Omega)}(W)$$

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Theorem (U.)

If $r(\Omega_a) < r(\Omega_b)$, then Ω_b cannot be contactly squeezed into Ω_a.

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Let W be a Liouville domain and let $P \subset \partial W$ be a contact polydisc in a contact Darboux chart. Then, the continuation map

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- Proof by analyzing the dynamics of contact Hamiltonians of the form

$$h(r, \theta, z) = \varepsilon + g(z) \cdot \prod f_j(r_j)$$

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- Not true for $\dim W = 2$!

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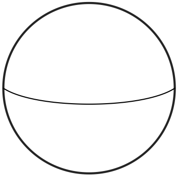

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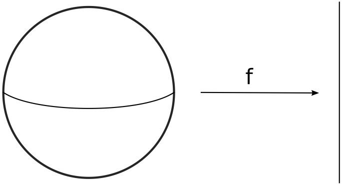

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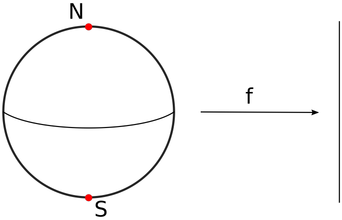

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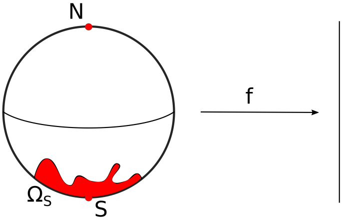

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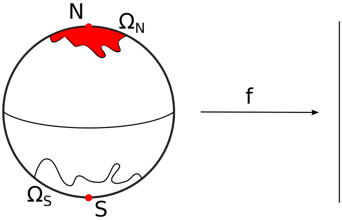

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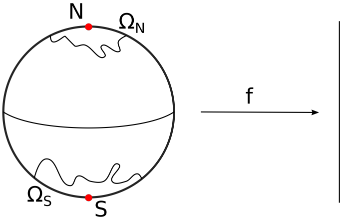

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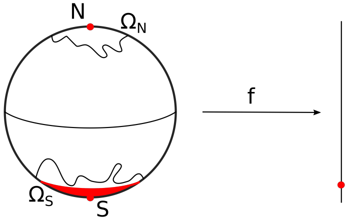

- $f^{-1}(-\infty, c] \subset \Omega_S$ for c close to $\min f$.

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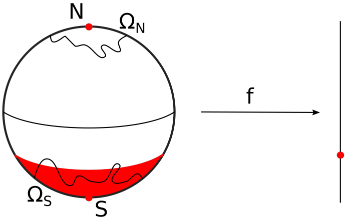

- $f^{-1}(-\infty, c] \subset \Omega_S$ for c close to $\min f$.

Proof of the non-squeezing

- Choose Ω_S such that $r(\Omega_S) < \infty$.
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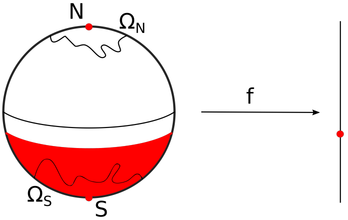

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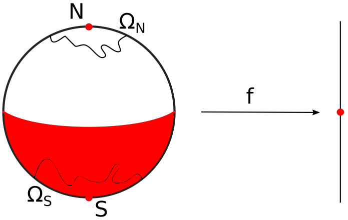

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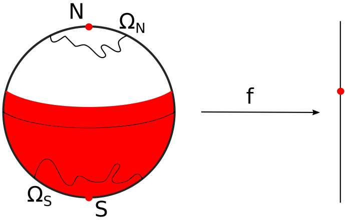

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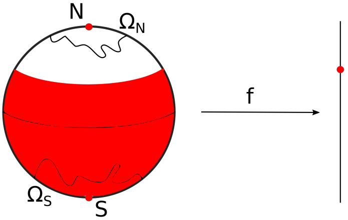

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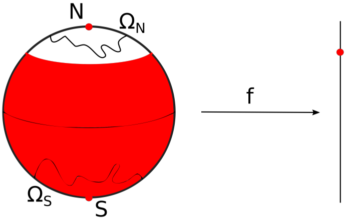

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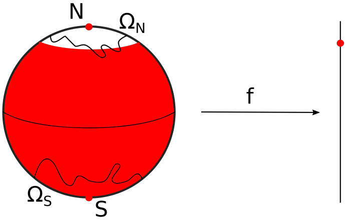

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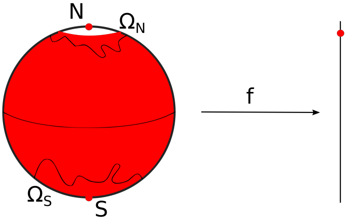

- $f^{-1}(-\infty, c] \subset \Omega_S$ for c close to $\min f$.
- $f^{-1}(-\infty, c] \supset \partial W \setminus \Omega_N$ for c close to $\max f$.

Thank you!