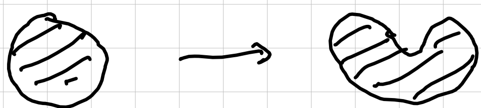
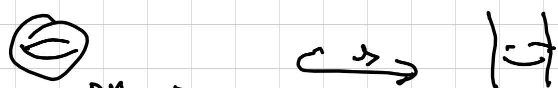
Symplectic convexity? (an ongoing story...)

In $(\mathbb{R}^{2n}, \omega_0 = \sum dx_i \wedge dy_i)$

Convex domains have very strong rigidity properties



$\underline{Q}$: What is the "right" symplectic notion



$B^{2n}(r) \overset{s}{\hookrightarrow} Z^{2n}(R) = D^2(R) \times \mathbb{R}^{2n-2}$

iff $r \leq R$

Symplectic capacities

$\underline{\text{Def}}$: A symp. cap. is a fct c which assigns to each symp mfd (X,ω) a number $c(X,\omega) \in [0,\infty]$ s.t.

1) $(X,\omega) \hookrightarrow (X',\omega') \Rightarrow c(X,\omega) \leq c(X',\omega')$

2) If $r \in \mathbb{R}_{>0} \Rightarrow c(X, r\omega) = r\,c(X,\omega)$

c is <u>ball normalized</u> if $c(B^{2n}(1)) = 1 = c(Z^{2n}(1))$

<u>Examples</u>: · Gromov width $c_{Gr}(X,\omega) = \sup\{r \mid \exists\, B^{2n}(r) \overset{s}{\hookrightarrow} X\}$

· cylindrical cap. $c_Z(X,\omega) = \inf\{R \mid \exists\, (X,\omega) \overset{s}{\hookrightarrow} Z^{2n}(R)\}$

- c_1^{EH}, c_{HZ}, c_{SH}, c_1^{CH}, ...

Conj (Viterbo): If $X \subset (\mathbb{R}^{2n}, \omega_0)$ is a convex set then for any ball. normalized symp cap.

$$(c(X))^n \leq n! \, Vol(X)$$

← euclidean vol

with "=" iff $X \simeq$ Ball

Lemma: If $X \subset \mathbb{R}^{2n}$ convex

$$(c_{Gr}(X))^n \leq n! \, Vol(X)$$

Thm (EH, HZ, AK, I, ...) If $X \subset \mathbb{R}^{2n}$ is convex

$$c_1^{EH}(X) = c_{HZ}(X) = c_{SH}(X) = c_1^{CH}(X) = sys(X)$$

Weak Viterbo conj

If $X \subset \mathbb{R}^{2n}$ convex

$$(sys(X))^n \leq n! \, Vol(X)$$

Strong Viterbo conj All ball. normalized symp cap. coincide on convex sets

Idea 1: being symplectomorphic to a convex set

Idea 2: HWZ Dynamical convexity

[convex $\Rightarrow$ dyn. convex]

Thm (ABHS) $\forall \varepsilon > 0$ $\exists$ a dyn. convex domain $C_\varepsilon \subset \mathbb{R}^4$ s.t. $(\mathrm{sys}(C_\varepsilon))^2 \geq (2-\varepsilon) \cdot 2! \,\mathrm{Vol}(C_\varepsilon)$

Thm (CE): $\exists$ $c, C > 0$ s.t. All convex sets K in $\mathbb{R}^4$ satisfy

$$c \leq \frac{\mathrm{sys}(K) \cdot \mathrm{Ru}(K)}{\mathrm{Vol}(K)} \leq C$$

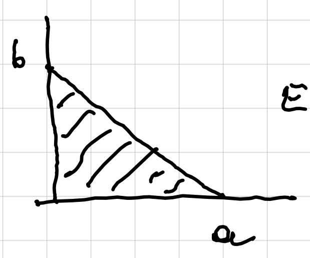
Thm (CE) Construction of a 4-dim dyn convex domain not symplecto to a convex domain

Idea 3: dyn. convex + toric

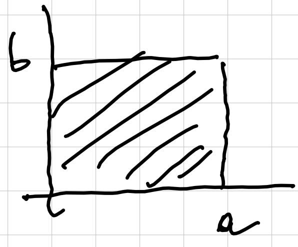
Def: A toric domain $X_\Omega \subset \mathbb{C}^n$ is a set of the form $X_\Omega = \mu^{-1}(\Omega)$

where $\cdot\ \Omega \subset [0,\infty)^n$
$\cdot\ \mu: \mathbb{C}^n \longrightarrow [0,\infty)^n : (z_1, \dots, z_n) \mapsto (\pi|z_1|^2, \dots, \pi|z_n|^2)$

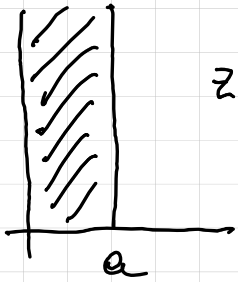
Ex:



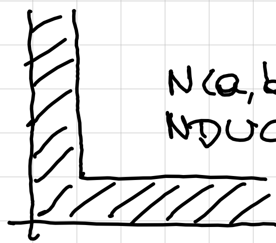
$E(a,b) = \{(z_1,z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\}$



$P(a,b)$

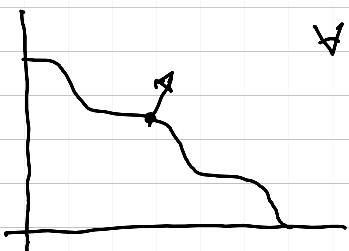


$Z^2(a)$



N(a,b)
NDUC

Def: A toric domain X_Ω is monotone if



$\forall x \in \partial_+ \Omega \left(= \partial\Omega \cap]0,\infty[^n\right)$

the normal vector to Ω at x

has all components non-neg

__Lemma__ (G. Hutchings. Ramos)

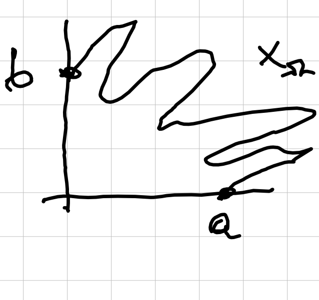
A toric domain $X_\Omega \subset \mathbb{R}^4$ is dyn. convex

iff X_Ω is (strict.) monotone

__Thm__ (G. H, R) All ball-normalized cap.

coincide on monotone toric domains in $\mathbb{R}^4$

__Lemma__ (H, Dardennes, G, Zhang)



$Ru(X_\Omega) = a+b$

<u>Thm</u> (D, G, Z) Any monotone toric domain can be deformed through monotone toric domains to a monotone toric domain which is not symplecto to a convex domain

Moreover this deformation is small w.r.t. symp. cap.

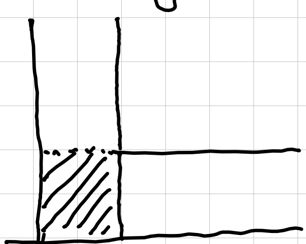
<u>Idea 4</u>: symp. convex is where all normalized cap. coincide

<u>Def</u>: The cube $D_n(\delta) = P(\delta, \dots, \delta)$

<u>Def</u>: (G, Hutchings) The cube capacity

$$c_D(X, \omega) = \sup\{\delta \mid \exists\, D_n(\delta) \hookrightarrow (X, \omega)\}$$

<u>Thm</u> (G, Hutching) $\quad c_D(N(\delta, \dots, \delta)) = \delta$



<u>Def</u>: (G, Pereira, Ramos) A cap. c is cube-normalized if $c(D_n(1)) = c(N(1, \dots, 1)) = 1$

Ex c_Z is cube-normalized

Not ball-normalized ($c_Z(B(1)) = \frac{1}{2}$, $c_Z(Z(1)) = 1$

Ex: Cieliebak-Mohnke

Let $L \subset (X, \omega)$ Lagrangian submfd

$A_{min}(L) := \inf\{\omega(\sigma) \mid \sigma \in \pi_2(X, L), \omega(\sigma) > 0\}$

Lagrangian capacity

$c_L(X, \omega) = \sup\{A_{min}(L) \mid L$ is an embedded Lag torus$\}$

Thm [Pereira] c_L is cube normalized

Thm [G, Pereira, Ramos] All cube-normalized symp. cap. coincide on all monotone toric domains in $\mathbb{R}^{2n}$

Conj cube-norm. cap. coincide on convex sets in $\mathbb{R}^{2n}$

A counterexample:

$X_a = \mu^{-1}(\Omega_a)$

$0 < a < \frac{1}{2}$

- $c_{Gr}(X_a) = \min\{1-a, 2-4a\}$
- $c_Z(X_a) = 1-a$
- $c_D(X_a) = \min\{1-2a, \frac{1}{2}\}$
- $c_L(X_a) = c^N(X_a) = \frac{1}{2}$

Conclusion :

$a \in]0, \frac{1}{4}[\longleftrightarrow$ cube-norm. cap. coincide

$a \in]0, \frac{1}{3}[\longleftrightarrow$ ball-norm cap coincide

$a \in]0, \frac{2}{5}[\longleftrightarrow c(X_a)^2 < 2 \cdot \mathrm{Vol}(X_a)$