The spectral diameter of a Liouville doimain and its applications

Pierre-Alexandre Mailhot

Université de Montréal

Symplectic Zoominar

October 28, 2022

The spectral norm and its diameter

(M, ω) a symplectic manifold. A *spectral invariant* is a function

$$c : H^*(M) \times C_c^\infty(S^1 \times M) \longrightarrow \mathbb{R}$$

that satisfies, for all $\beta, \eta \in H^*(M)$ and $H, K \in C_c^\infty(S^1 \times M)$,

- [Continuity] $|c(\beta, H) - c(\beta, K)| \leq \|K - H\|$ where

$$\|F\| = \int_0^1 \left(\sup_{p \in M} F(t,p) - \inf_{p \in M} F(t,p) \right) \mathrm{d}t,$$

- [Non-degenerate spectrality] $c(\beta, H) \in \mathrm{Spec}(H)$ for non-degenerate H,
- [Triangle inequality] $c(\beta \cup \eta, H\#K) \leq c(\beta, H) + c(\eta, K)$, where $H\#K(t,p) = H(t,p) + K(t, (\varphi_H^t)^{-1}(p))$.

Spectral invariants are known to exist in the following settings

1. $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$, Viterbo 1992.
2. Closed symplecticaly aspherical manifolds, Schwarz 2000.
3. Closed symplectic manifolds, Oh 2005. See also Usher 2013.
4. Convex symplectic manifolds, Frauenfelder and Schlenk 2007.

In case 3 above, to define c, we need to take into account quantum phenomena and instead have a function

$$c : QH^*(M) \times C^\infty(S^1 \times M) \longrightarrow \mathbb{R}$$

- Schwarz proved that if $\varphi_H = \varphi_K \in \mathrm{Ham}_c(M)$, then

$$c(1,H) = c(1,K)$$

$$\implies \quad c(1,\varphi) := c(1,H) \text{ for } \varphi = \varphi_H \text{ is well defined}$$

- The *spectral norm* $\gamma : \mathrm{Ham}_c(M) \to \mathbb{R}$ is defined as

$$\gamma(\varphi) = c(1,\varphi) + c(1,\varphi^{-1}) = c(1,H(t,p)) + c(1,-H(t,\varphi_H^t(p))).$$

- Can prove that

$$\gamma(\varphi) \leq \nu_{\mathrm{Hofer}}(\varphi) =: \inf\{\|H\| \mid \varphi = \varphi_H\}$$

- It is thus natural to ask whether the spectral diameter

$$\mathrm{diam}_\gamma(M) = \sup\{\gamma(\varphi) \mid \varphi \in \mathrm{Ham}_c(M)\}$$

is finite or not.

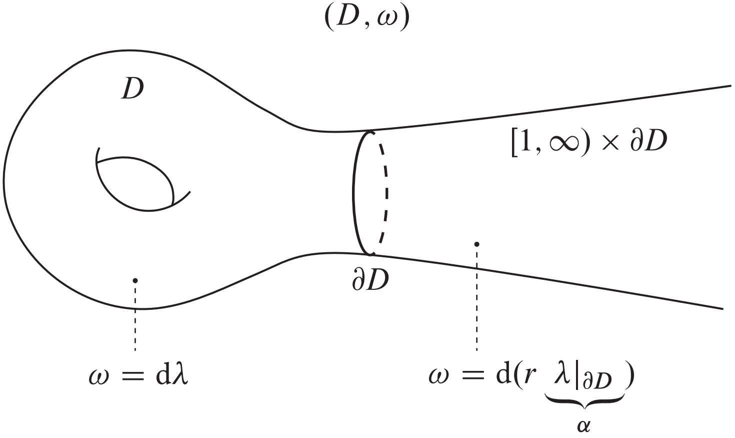
- For a surface Σ_g of genus $g > 1$, $\mathrm{diam}_\gamma(\Sigma_g) = +\infty$.
- $\mathrm{diam}_\gamma(S^2, \omega) \leq \omega(S^2)$.
- More generally,

$$\mathrm{diam}_\gamma(\mathbb{C}P^n, \omega_{\mathsf{FS}}) = \frac{n}{n+1} \int_{\mathbb{C}P^1} \omega_{\mathsf{FS}},$$

  Entov-Polterovich 2003 and Kislev-Shelukhin 2018.
- (M, ω) with $H \in C^\infty(M)$ such that all its contractible orbits are constant, then $\mathrm{diam}_\gamma(M) = +\infty$, Kislev-Shelukhin 2018.
- For DT^*N the unit cotangent disk bundle over closed N, $\mathrm{diam}_\gamma(DT^*N) = +\infty$, Monzner-Vichery-Zapolsky 2012.

We will now study the finiteness of diam_γ for Liouville domains.

Liouville domains


$(\hat{D}, \omega)$
D
$[1, \infty) \times \partial D$
∂D
$\omega = \mathrm{d}\lambda$
$\omega = \mathrm{d}(r \underbrace{\lambda|_{\partial D}}_{\alpha})$

- Given an Hamiltonian H which is linear outside a compact set in $\hat{D}$, can define

$$HF^*_{(a,c)}(H)$$

$$\longrightarrow HF^*_{(a,b)}(H) \xrightarrow{[\iota_a^{b,c}]} HF^*_{(a,c)}(H) \xrightarrow{[\pi^c_{a,b}]} HF^*_{(b,c)}(H) \longrightarrow$$

- We can extend a compactly supported H on D to an Hamiltonain H^ε linear at infinity with small slope ε and define its Floer cohomology as

$$HF^*_{(a,c)}(H) = HF^*_{(a,c)}(H^\varepsilon).$$

- $HF^*(H)$ is isomorphic to $H^*(D)$ from which it inherits a unit 1 for the pair of pants product.

- The *spectral invariant of H* is then defined as

$$c(1, H) = \inf\{c \in \mathbb{R} \mid 1 \in \operatorname{im} \iota^{<c}\}$$

  where $\iota^{<c} = \iota_{-\infty}^{c,+\infty}$.

- Choose a sequence $\{H_i\}_{i \in I}$ of Hamiltonians linear at infinity such that $H_i|_D \leq \cancel{D}\ 0$ with slope $\to \infty$ as $i \to \infty$. The *filtered symplectic cohomogoly of D* is defined as

$$SH^*_{(a,b)}(D) = \varinjlim_{H_i} HF^*_{(a,b)}(H_i)$$

- For small enough $\varepsilon > 0$,

$$SH^*_{(-\infty,\varepsilon)}(D) \cong H^*(D).$$

- It is already known that if $SH^*(D) = 0$, then there exists a uniform bound on all spectral invariants on D
- Define

$$c_{SH}(D) = \inf\{c > 0 \mid [\iota_{-\infty}^{\varepsilon,c}] = 0\} \in (0, \infty].$$

It corresponds to the action level at which $H^*(D)$ vanishes in $SH^*(D)$. Then, $c_{SH}(D)$ is finite $\iff$ $SH^*(D) = 0$.

Theorem (Benedetti-Kang 2020)

Suppose $SH^*(D) = 0$. Then,

$$\sup_H\{c(1, H)\} \leq c_{SH}(D) < +\infty.$$

Main results

- If $SH^*(D) = 0$, by the previous Theorem, the spectral norm therefore satisfies the bound

$$\gamma(H) = c(1, H) + c(1, \overline{H}) < 2c_{SH}(D) < +\infty.$$

- It remains to find when exactly diam_γ is infinite. We prove the

Theorem (M. 2022)

If $SH^*(D) \neq 0$, *then* $\mathrm{diam}_\gamma(D) = +\infty$

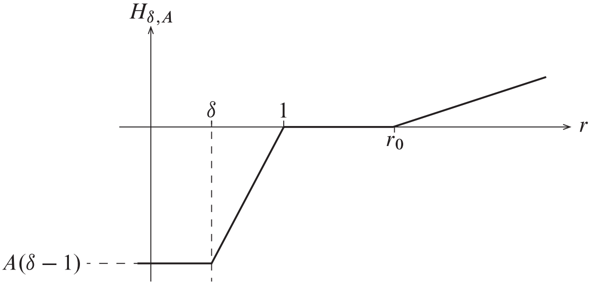
(1) Construct an Hamiltonian H with $c(1, H)$ arbitrarily large.

(2) Show that $c(1, \overline{H}) \geq 0$. This relies on the

Lemma (Ganor-Tanny 2020, M. 2022)

If K is compactly supported in D, then $c(1, K) \geq 0$.

- Fix $A \in (0,\infty) \setminus \mathrm{Spec}(\partial D, \lambda)$ and let η_A be the distance between A and $\mathrm{Spec}(\partial D, \lambda)$.
- We can choose $0 < \delta < 1$ and $\varepsilon > 0$ so that $\delta A < \varepsilon < \eta_A$.



- In terms of action of orbits, $(III) < A - \varepsilon < (I) < (II)$.

Claim : $c(1, H_{\delta,A}) \geq A - \varepsilon$

- Since $(III) < A - \varepsilon < (I) < (II)$, we have the complexes

$$C^*_{III} = CF^*_{<A-\varepsilon}(H_{\delta,A}), \quad C^*_{I,II} = \frac{CF^*(H_{\delta,A})}{C^*_{III}} = CF^*_{(A-\varepsilon,\infty)}(H_{\delta,A}).$$

- Build maps Ψ and $\Psi_{I,II}$ so that we have a commutative diagram :

$$\begin{tikzcd} HF^*(H_{\delta,A}) \arrow[r, "{[\pi_{>A-\varepsilon}]}"] \arrow[dr, "\Psi"'] & H^*(C^*_{\mathrm{I,II}}) \arrow[d, "\Psi_{\mathrm{I,II}}"] \\ & SH^*_{(-\varepsilon,\infty)}(D) \end{tikzcd}$$

- Here, Ψ needs to coincide with the Viterbo map $j_{H_{\delta,A}}$. That way, it will be a map of unital algebras.

- By construction and commutativity,

$$\Psi(1_{H_{\delta,A}}) = 1_D = [\pi_{>A-\varepsilon}] \circ \Psi_{\mathrm{I,II}}(1_{H_{\delta,A}})$$

- Thus, $[\pi_{>A-\varepsilon}](1) \neq 0$ and from the long exact sequence in cohomology

$$\longrightarrow HF^*_{<A-\varepsilon}(H) \xrightarrow{[\iota^{<A-\varepsilon}]} HF^*(H) \xrightarrow{[\pi_{>A-\varepsilon}]} HF^*_{>A-\varepsilon}(H) \longrightarrow$$

  we have

$$1 \notin \mathrm{im}[\iota^{<A-\varepsilon}] \quad \Longrightarrow \; c(1, H_{\delta,A}) \geq A - \varepsilon.$$

- Following a continuity argument, we can use $H_{\delta,A}$ to precisely compute the spectral invariant of many Hamiltonians.

Lemma (M. 2022)

Suppose $SH^*(D) \neq 0$. Let H compactly supported autonomous such that, for $A > 0$,

$$H\big|_{sk(D)} = -A \quad \textit{and} \quad -A \leq H\big|_{D} \leq 0.$$

Then, $c(1, H) = A$.

- The main Theorem follows from a sharper result.
- Using the previous computation, we can build, when $SH^*(D) \neq 0$, we can build an explicit isometric group embedding

$$(\mathbb{R}, d_{\mathrm{st}}) \to (\mathrm{Ham}_c(D), d_\gamma)$$

where $d_\gamma(\varphi, \psi) = \gamma(\varphi \circ \psi^{-1})$ and d_{st} is the standard Euclidian distance on $\mathbb{R}$.

Applications

First some definitions

- (M, ω) is symplectically aspherical is ω and the first Chern class $c_1(M)$ both vanish on $\pi_2(M)$.
- An open subset $U \subset M$ is incompressible if $\pi_1(U) \to \pi_1(M)$ is injective.

Then, we have the

Proposition (M. 2022)

- *(M, ω) symplectically aspherical*
- *D incompressible Liouville domain of codimension 0 embedded inside M with $SH^*(D) \neq 0$.*

$$\implies \ diam_\gamma(M) = +\infty.$$

To prove this Proposition, we proceed as follows:

- Show that, for H compactly supported on D, $c_M(1_M, H) = c_D(1_D, H)$
- Use the main theorem.

From the Proposition, we can directly deduce the

Corollary

Let (M, ω) be closed and symplecticaly aspherical. Then,

$$diam_\gamma(M \times M, \omega \oplus -\omega) = +\infty.$$

- For any $A > 0$, define

$$E_A(M,\omega) := \{\varphi \in \mathrm{Ham}(M,\omega) \mid d_H(\mathrm{Id},\varphi) > A\}\,.$$

In 2010, LeRoux posed the following question

Does $E_A(M,\omega)$ have non-empty C^0-interior for all $A > 0$?

Theorem (Buhovsky-Humilière-Seyfaddini 2021)

- (M,ω) *closed, connected and symplectically aspherical,*
- $\mathrm{diam}_\gamma(M) = +\infty$.

$\implies$ $E_A(M,\omega)$ *has non-empty C^0-interior for all $A > 0$.*

The previous Theorem and Corollary can be used to give a partial answer to the question of LeRoux.

Corollary (M. 2022)

- (M, ω) *closed, connected and symplectically aspherical,*

$\Longrightarrow$ $E_A(M \times M, \omega \oplus -\omega)$ *has non-empty* C^0*-interior for all* $A > 0$.

Thank you :-)