Obstructions and constructions for staircases in Hirzebruch surfaces

Including work with Dusa McDuff, Ana Rita Pires, and Morgan Weiler

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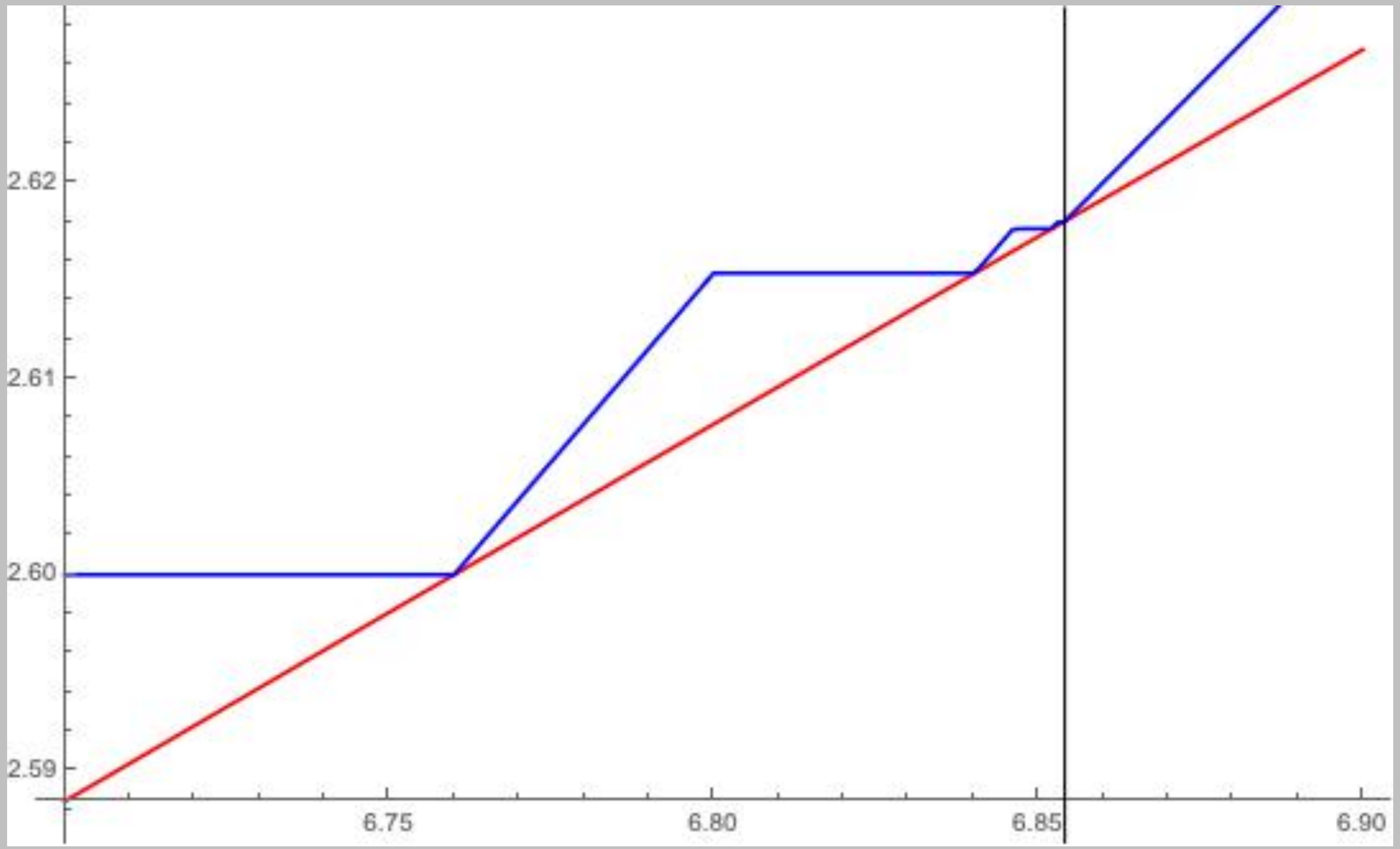
Summary of Main Result

We give a complete classification to which Hirzebruch surfaces have a property called an infinite staircase.



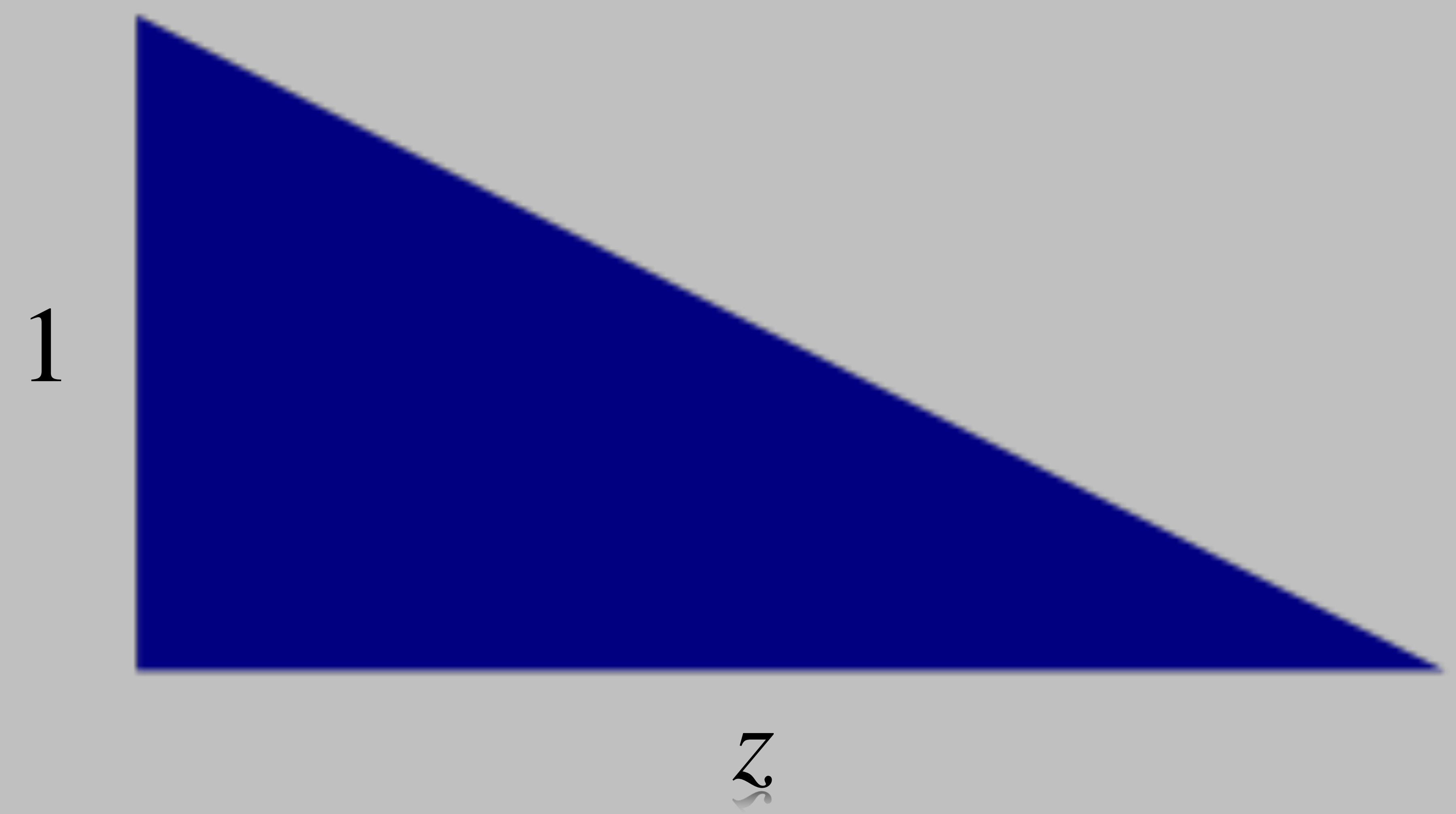
Symplectic Embeddings

A **symplectic embedding** $\varphi : (X_1, \omega_1) \overset{s}{\hookrightarrow} (X_2, \omega_2)$ is a smooth embedding such that $\varphi^*(\omega_2) = \omega_1$.

If $\varphi : (X_1, \omega_1) \overset{s}{\hookrightarrow} (X_2, \omega_2)$, then $vol(X_1) \leq vol(X_2)$.

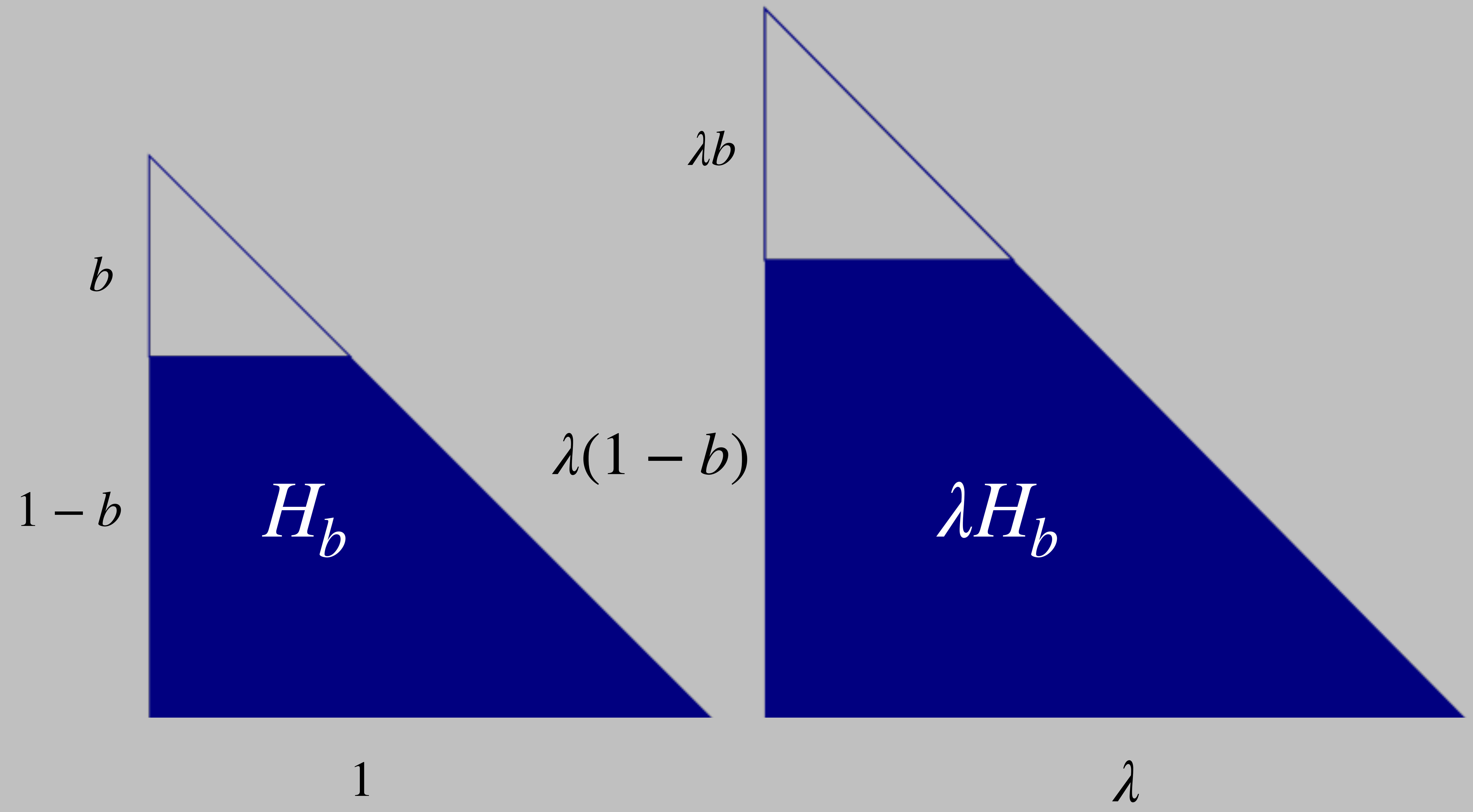
Domain of Embeddings: Ellipsoids

$E(1, z)$ **is the preimage of**



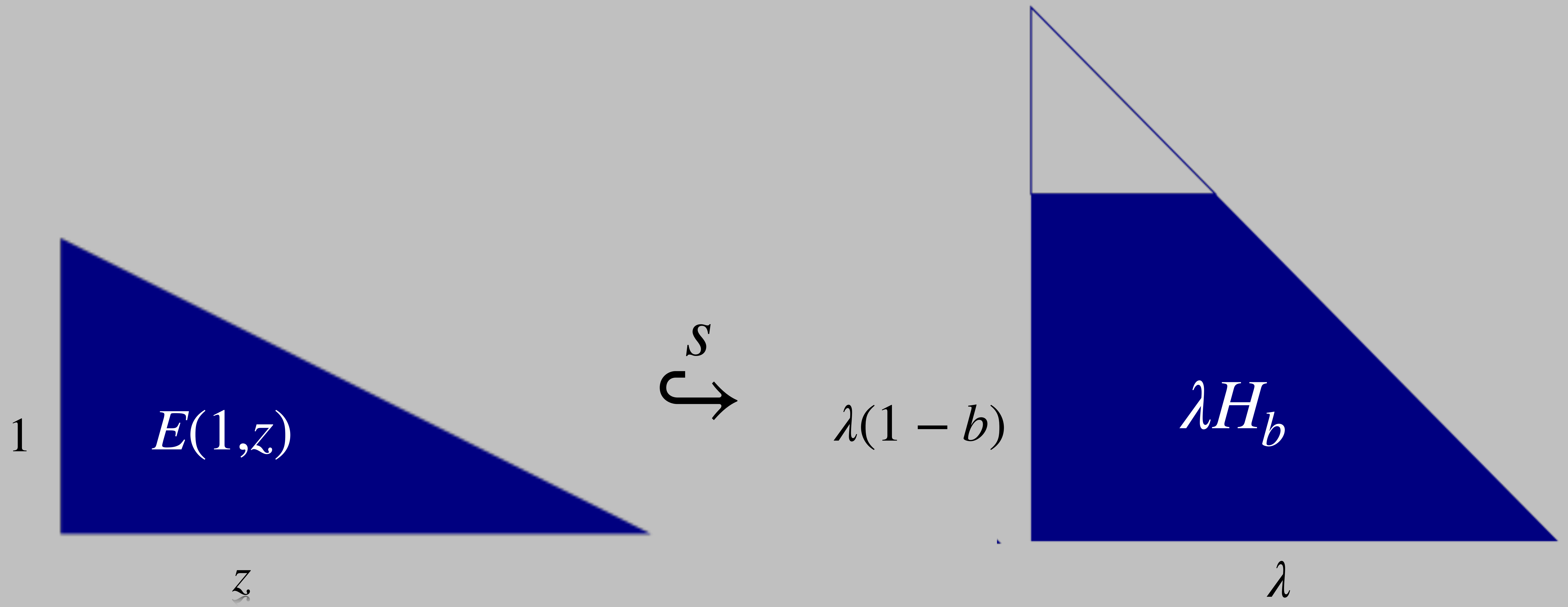
under the map from $\mathbb{C}^2 \to \mathbb{R}^2$ **where** $(z_1, z_2) \mapsto (\pi |z_1|^2, \pi |z_2|^2)$

Targets of Embeddings: $\mathbb{C}P^2_1 \# \overline{\mathbb{C}P}^2_b$



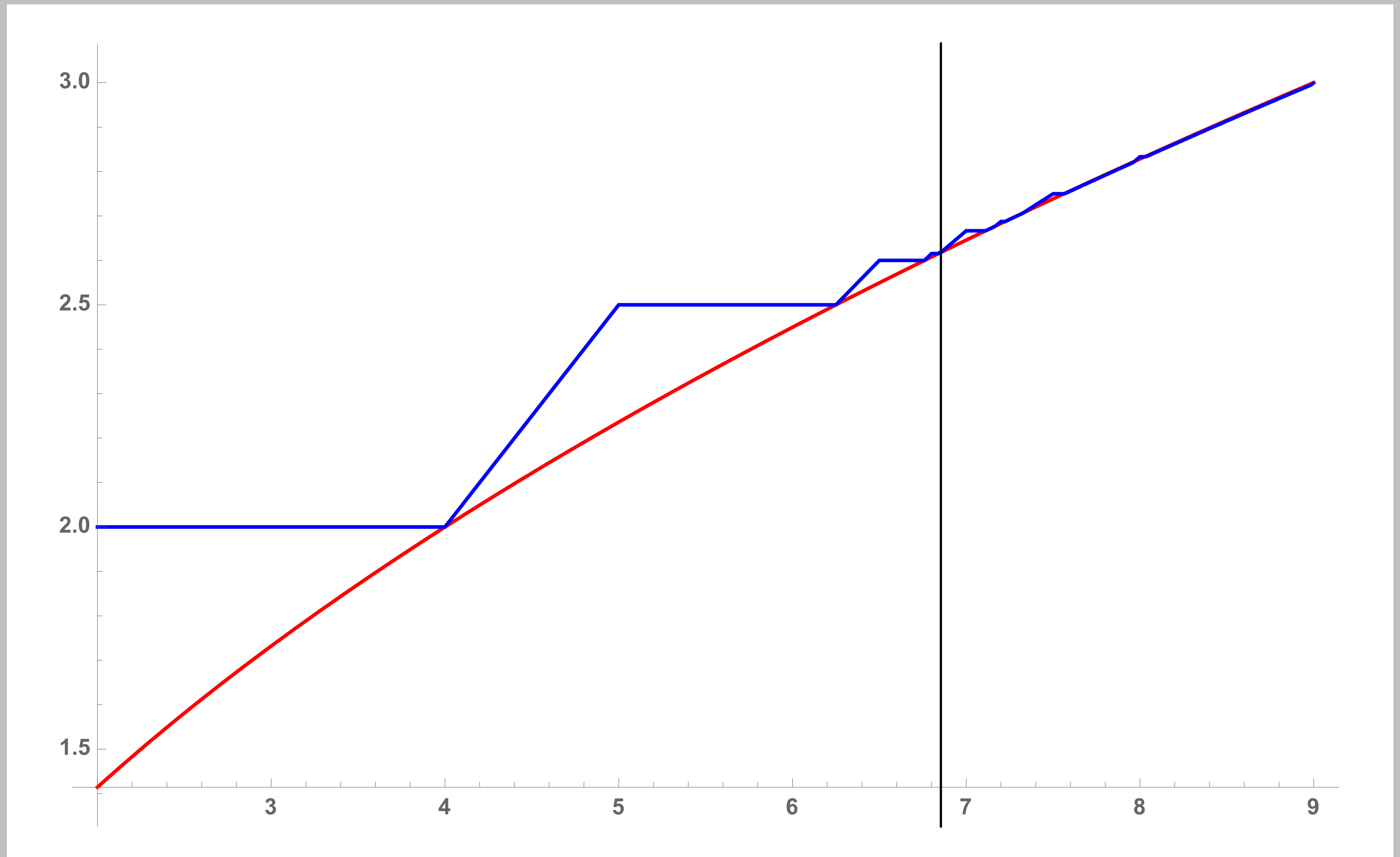
Main Question

For what λ, does $E(1,z) \overset{s}{\hookrightarrow} \lambda H_b$?

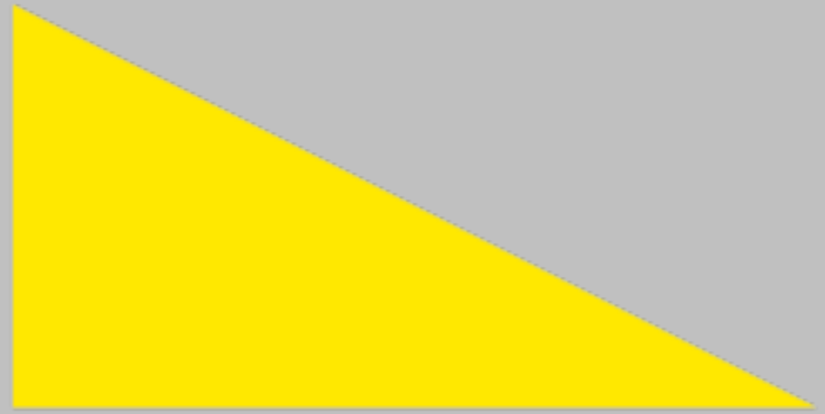


Object of Study

Embedding function: $c_b(z) := \inf\{\lambda \mid E(1,z) \overset{s}{\hookrightarrow} \lambda H_b\}$

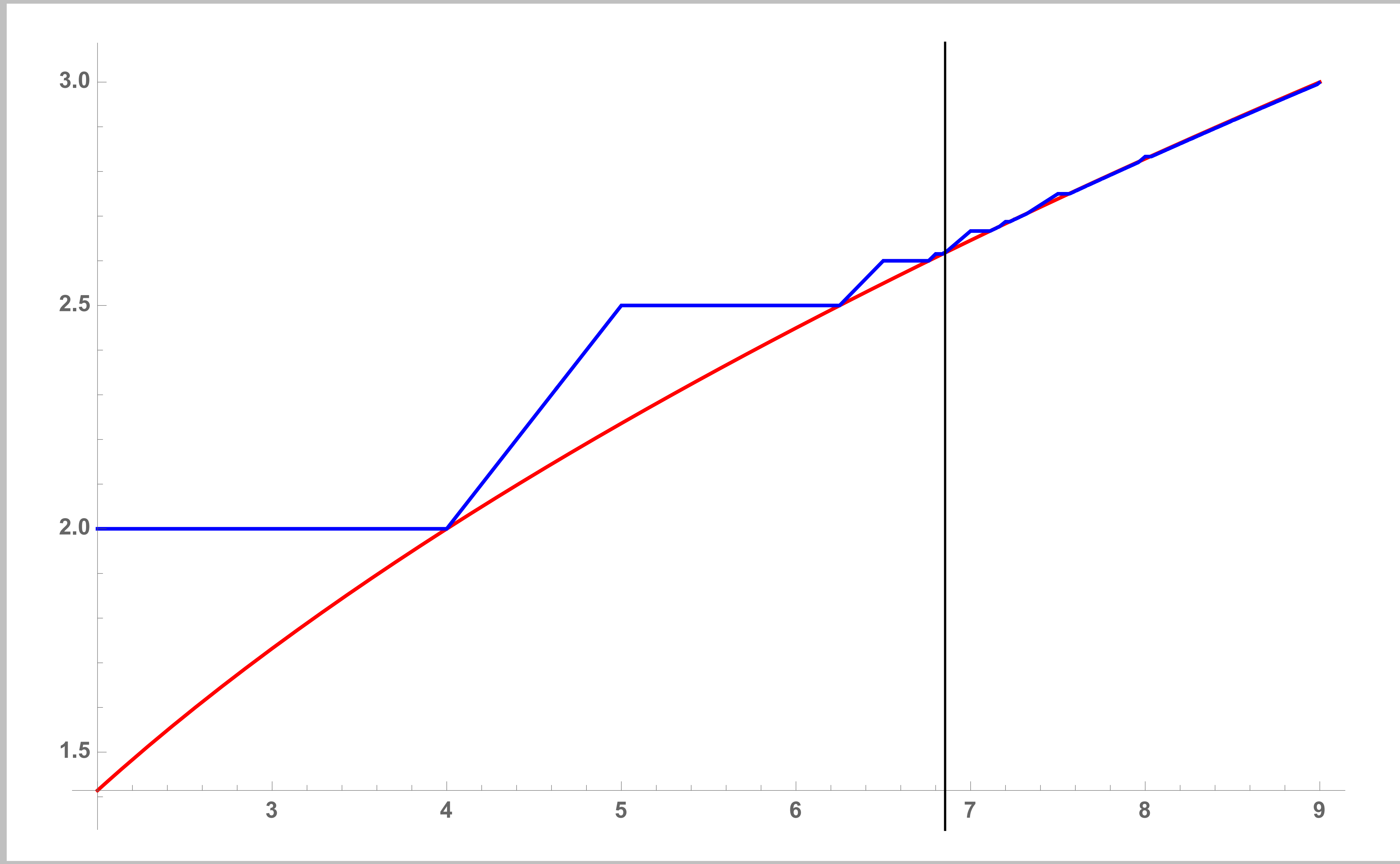


Minimum scaling of 

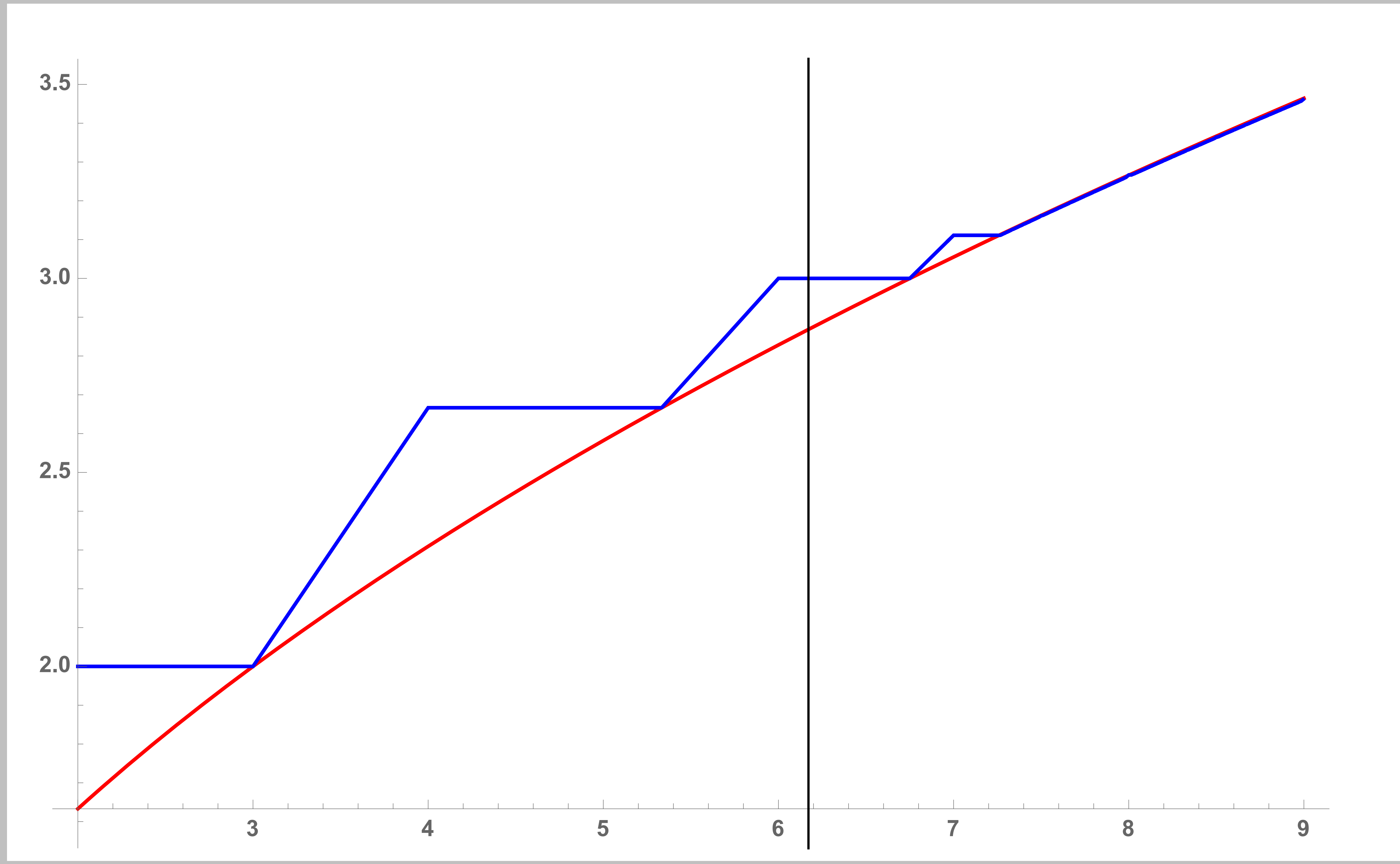
Size E(1,z) of ellipsoid 

Properties of Embedding Function

(Cristofaro Gardiner-Holm-Mandini-Pires, 2020)



Infinite Staircase (infinitely many non smooth points)



No Infinite Staircase (finitely many non smooth points)

The Main Question

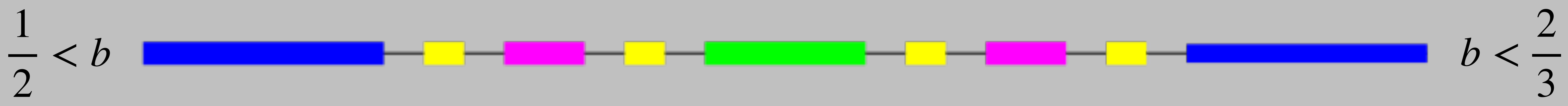
Which $b \in [0,1)$ values does the embedding function $c_b(z)$ have an infinite staircase?

- McDuff-Schlenk (2010) showed $b = 0$ has an infinite staircase.
- Cristofaro Gardiner-Holm-Mandini-Pires (2020) showed $b = \frac{1}{3}$ has an infinite staircase and conjectured this is the only rational value other than 0 with an infinite staircase.
- Bertozzi-Holm-Maw-McDuff-Mwakyoma-Pires-Weiler (2021) found infinitely many irrational b values with infinite staircases.

Block and Stair

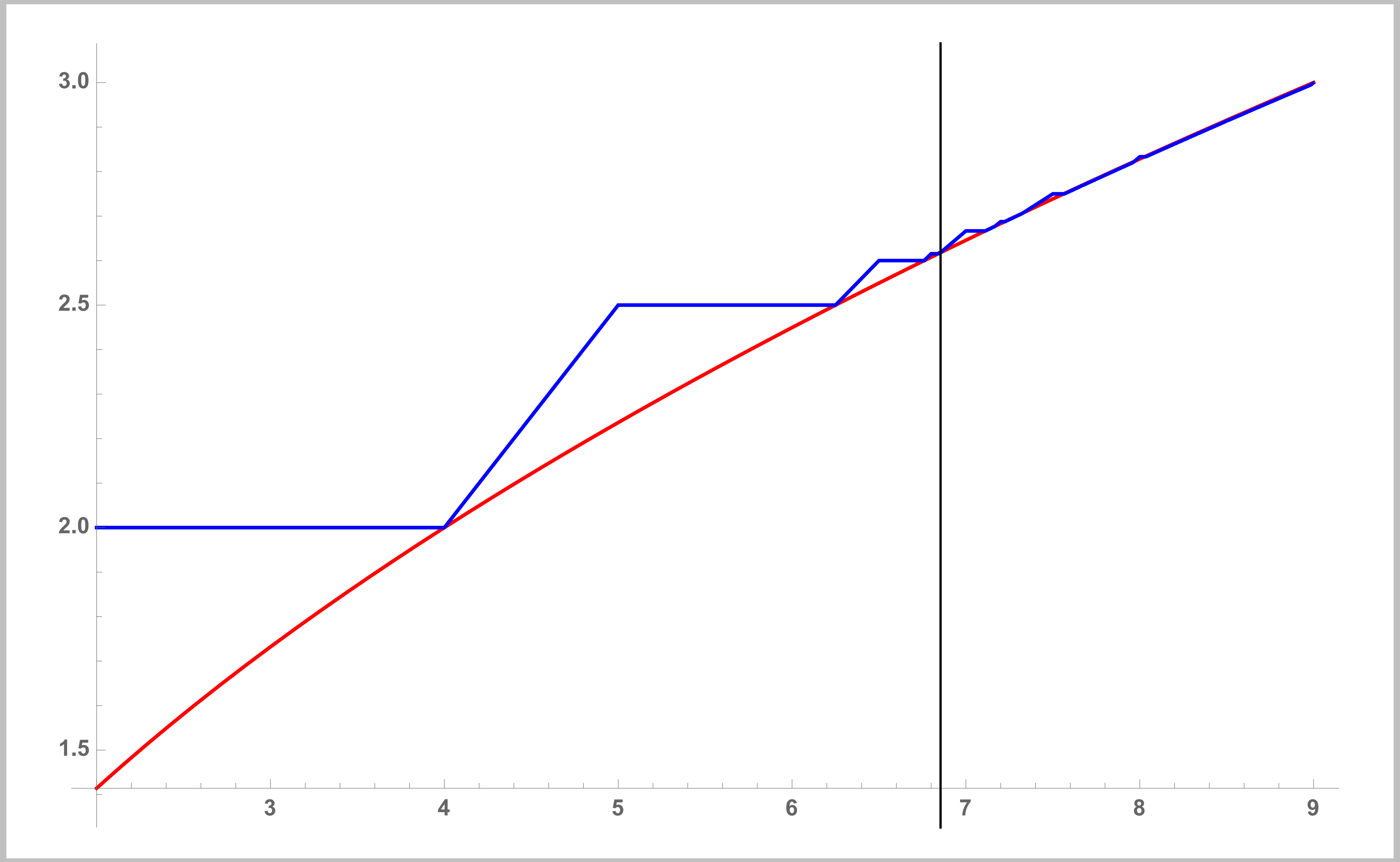
M., McDuff, Weiler

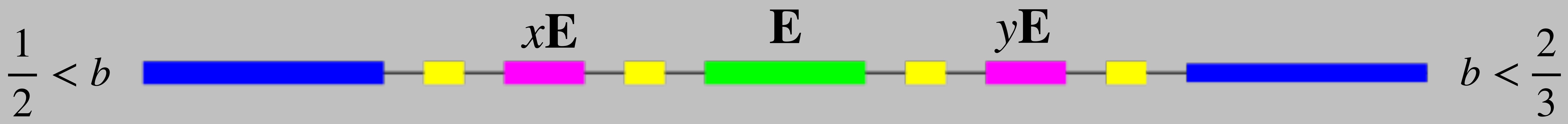
- Block is a certain subset of b with no infinite staircases.
- Block is an open dense set in (0,1).
- For each n, Block $\cap \left(\frac{n}{n+1}, \frac{n+1}{n+2}\right)$ is homeomorphic to the complement of the middle third Cantor set.
- There are infinite staircases at the endpoints of these blocked intervals.



The Focus of the Talk:
M.

For the b-values at the left endpoint of blocked intervals, for the possible accumulation point z_b, we have $c_b(z_b) = vol(z_b)$.





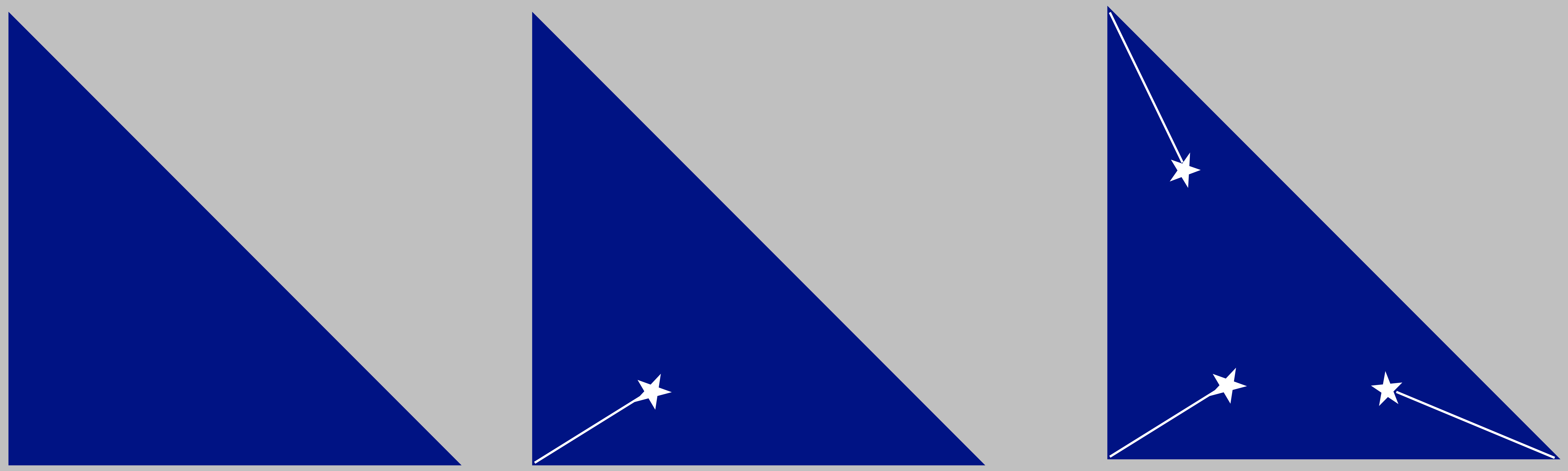
- Each interval is described by a homology class $\mathbf{E} \in H_2(\mathbb{C}P^2\#_k\overline{\mathbb{C}P}^2, \mathbb{Z})$ representing a symplectic sphere of self intersection -1. McDuff-Polterovich and Li-Li implies all non-volume obstructions to embeddings can be described this way.

- In [M.MW], we constructed a mutation process on the homology classes $x\mathbf{E}$ and $y\mathbf{E}$ given by recursive formulas.

- If we have an interval I, the classes that give the steps of the staircase that accumulate at the left endpoint of the interval are given by $y^k\mathbf{E}$ for some $\mathbf{E}$.

Almost Toric Fibrations

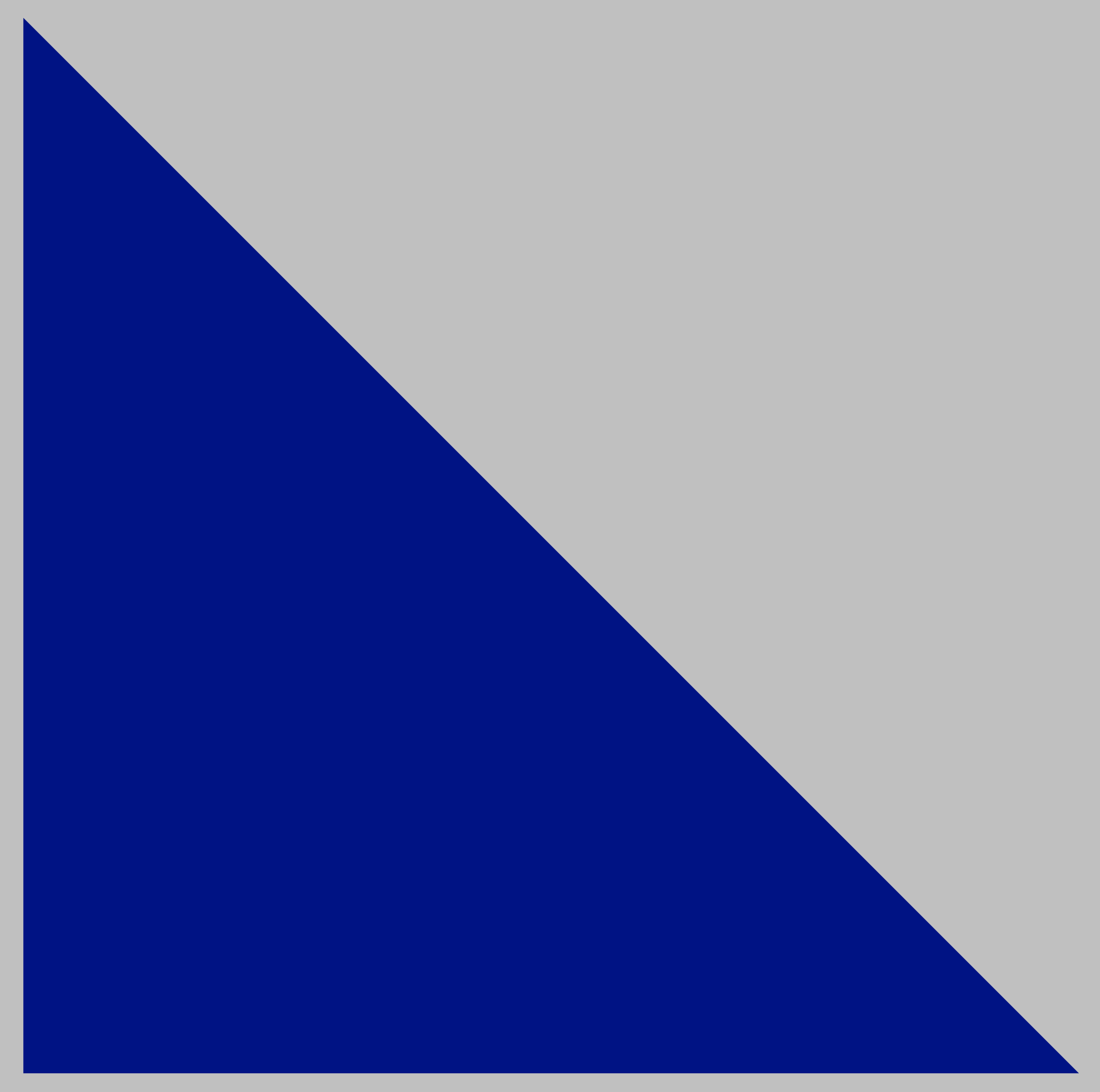
Symington and Leung classified closed almost toric manifolds in terms of the base diagrams with decorations for the various singularities.



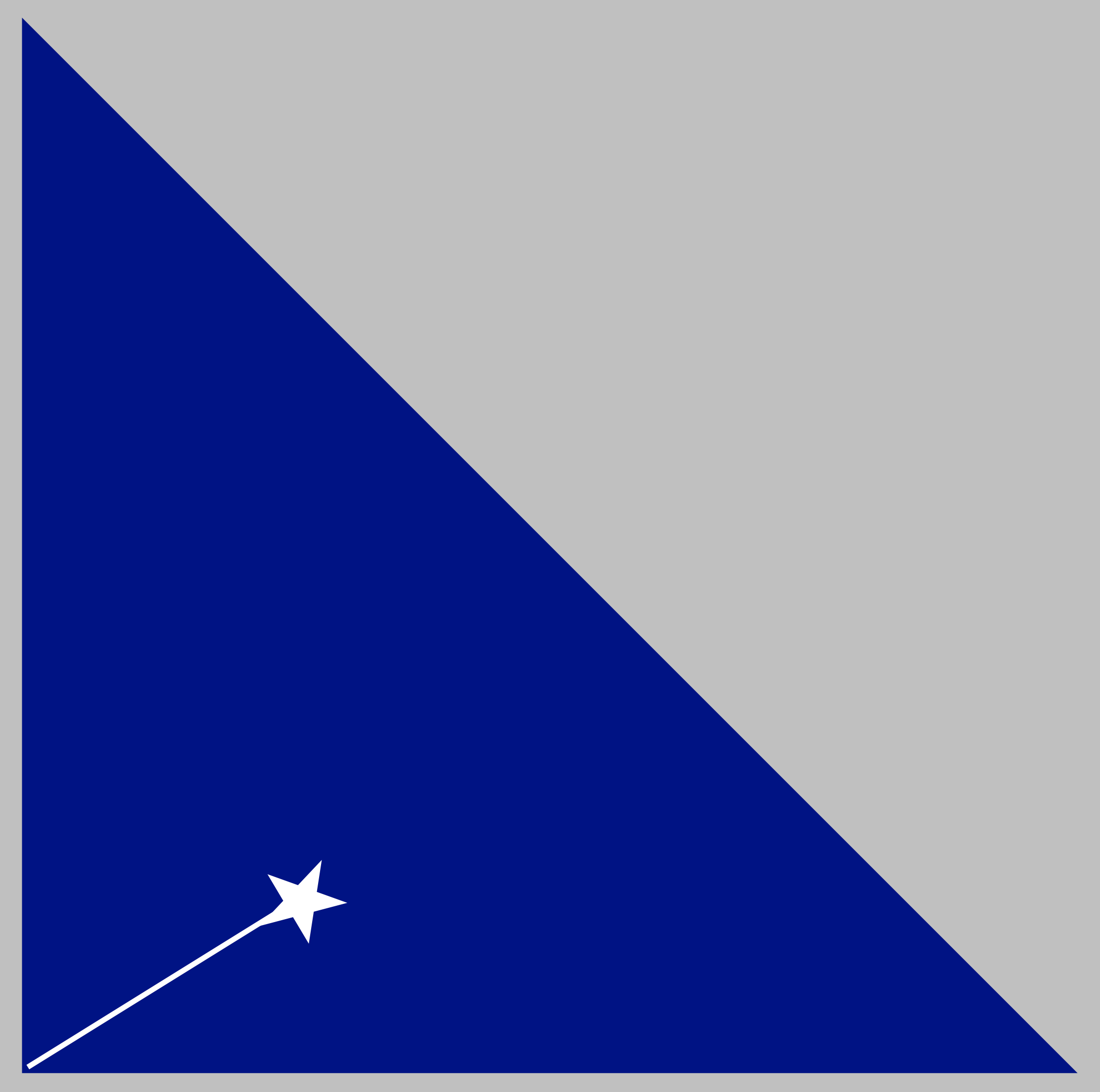
$$\mathbb{C}P^2 \to \mathbb{R}^2$$

$$[z_0 : z_1 : z_2] \mapsto \left(\frac{|z_0|^2}{|z|^2}, \frac{|z_1|^2}{|z|^2} \right)$$

Two different fibrations for symplectomorphic $\mathbb{C}P^2$



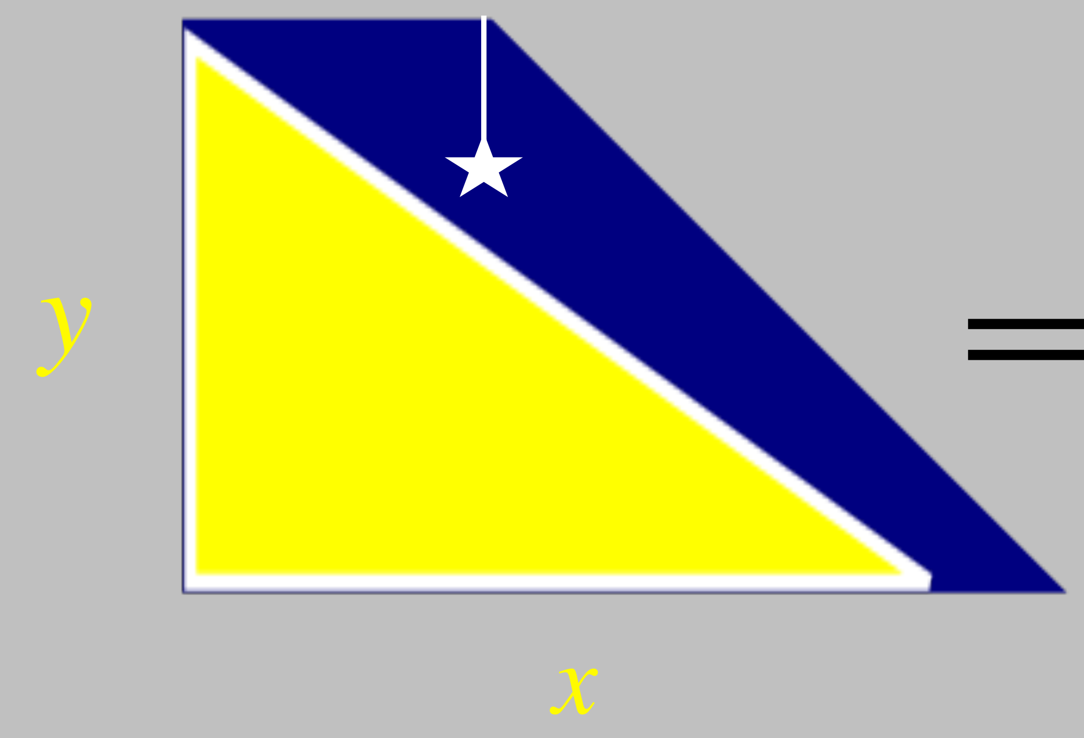
- Near origin, looks like $z_0 z_1 = 0$
- Fiber over interior points are Lagrangian tori
- Fiber over interior of an edge is a circle
- Fiber over vertices are points



- Near origin, looks like $z_0 z_1 = c$
- Fiber over star is pinched torus
- Fiber over the ray vertex is a circle
- Nodal ray is a branch cut whose direction is eigenvector of monodromy

Almost Toric Pictures and Embeddings

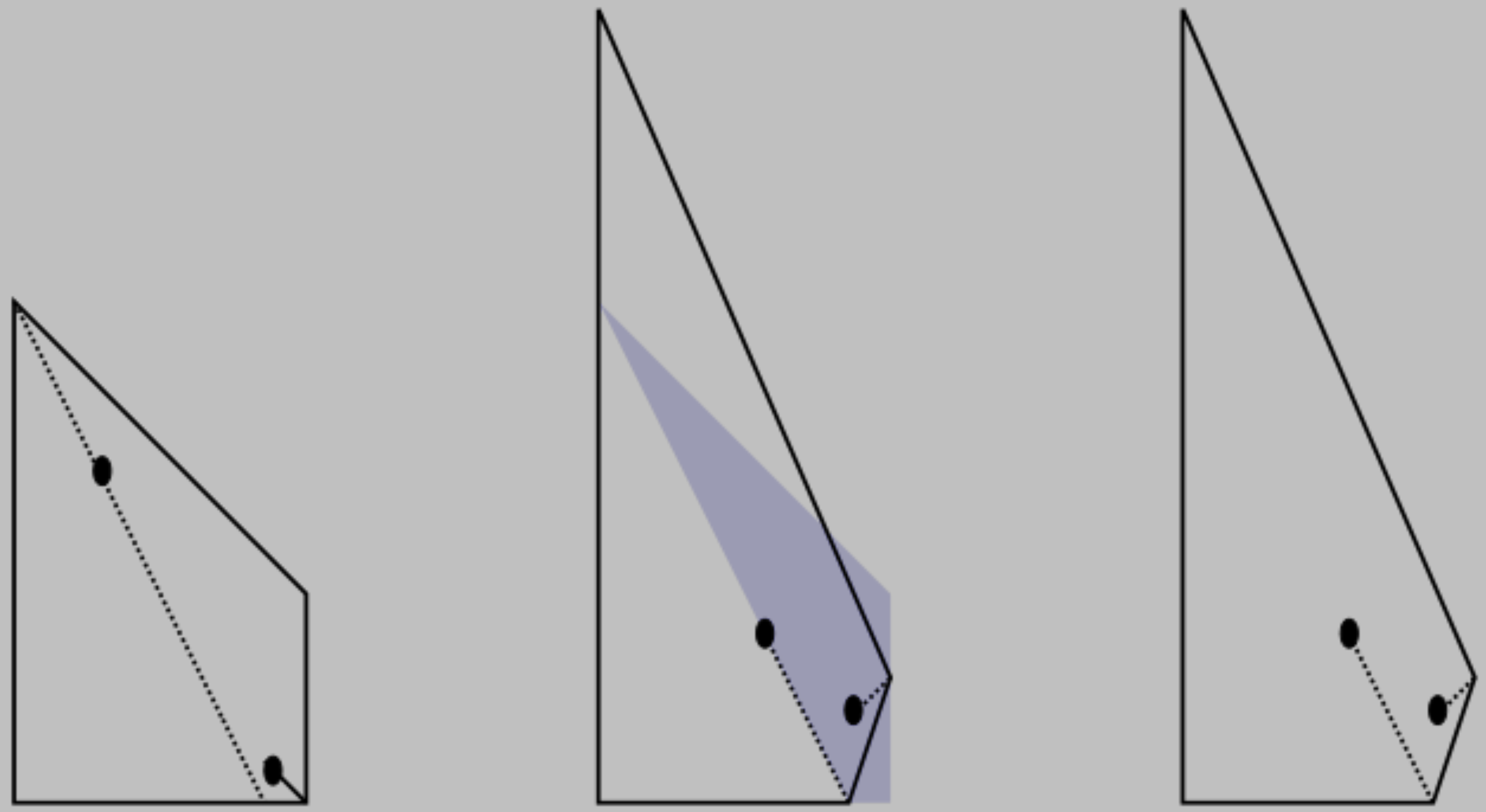
Idea used in Casals-Vianna and CG-HMP



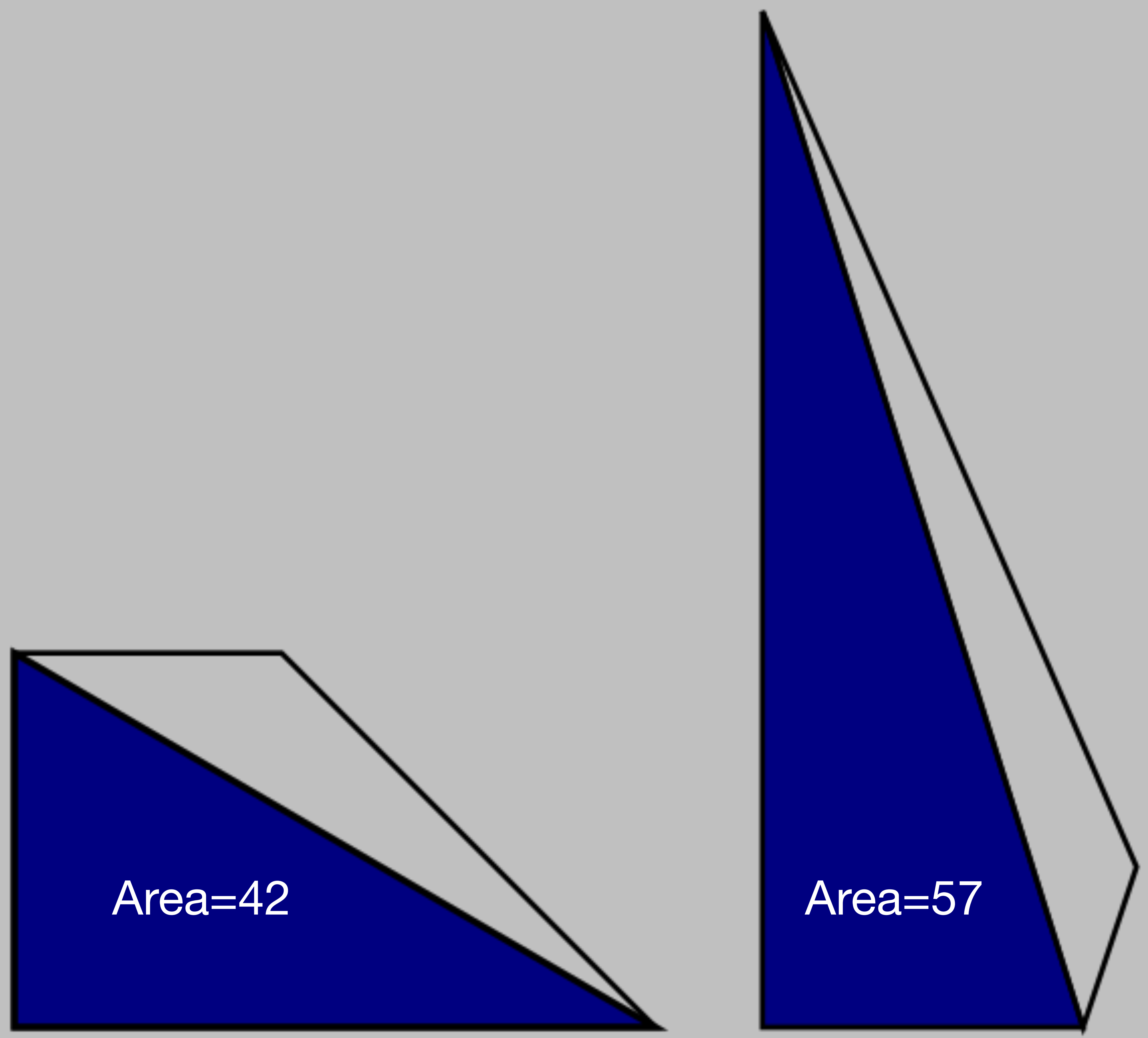
$$\Longrightarrow (1-\epsilon)E(x,y) \overset{s}{\hookrightarrow} H_b \Longrightarrow c_b\left(\frac{y}{x}\right) \leq \frac{1}{x}$$

Embeddings gives upper bounds of the embedding function

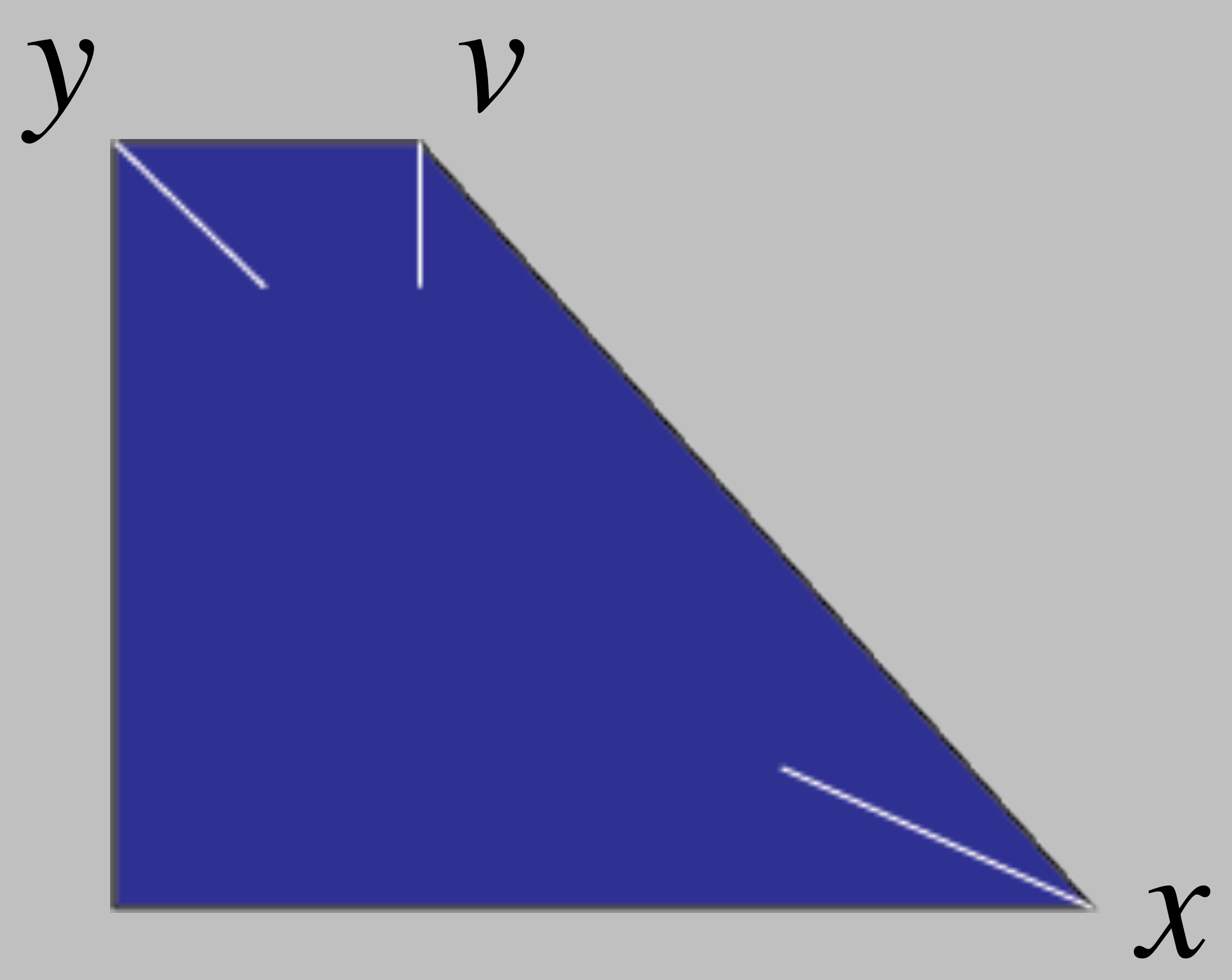
Almost Toric Fibration Mutations



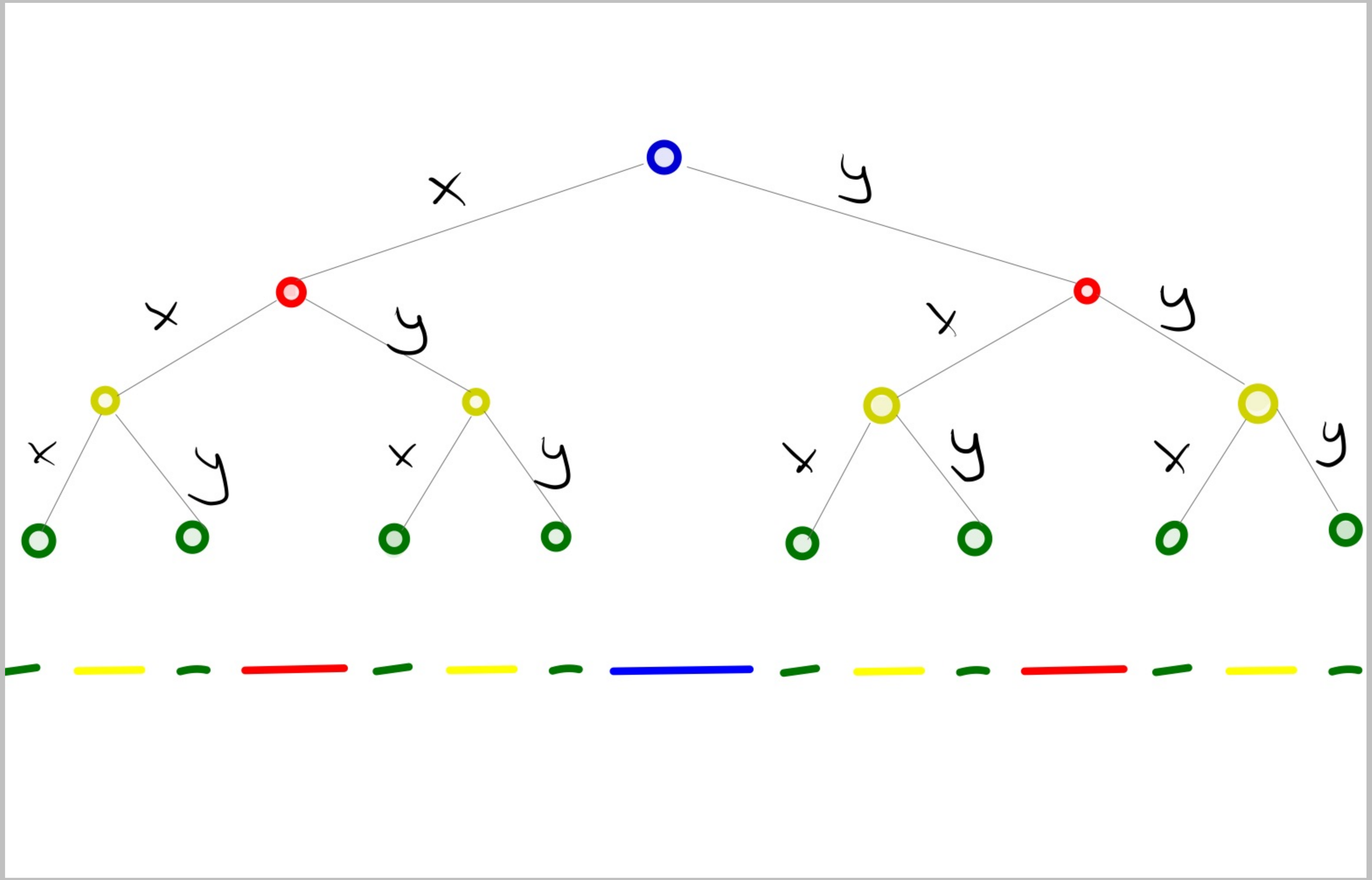
Almost Toric Fibration Mutations

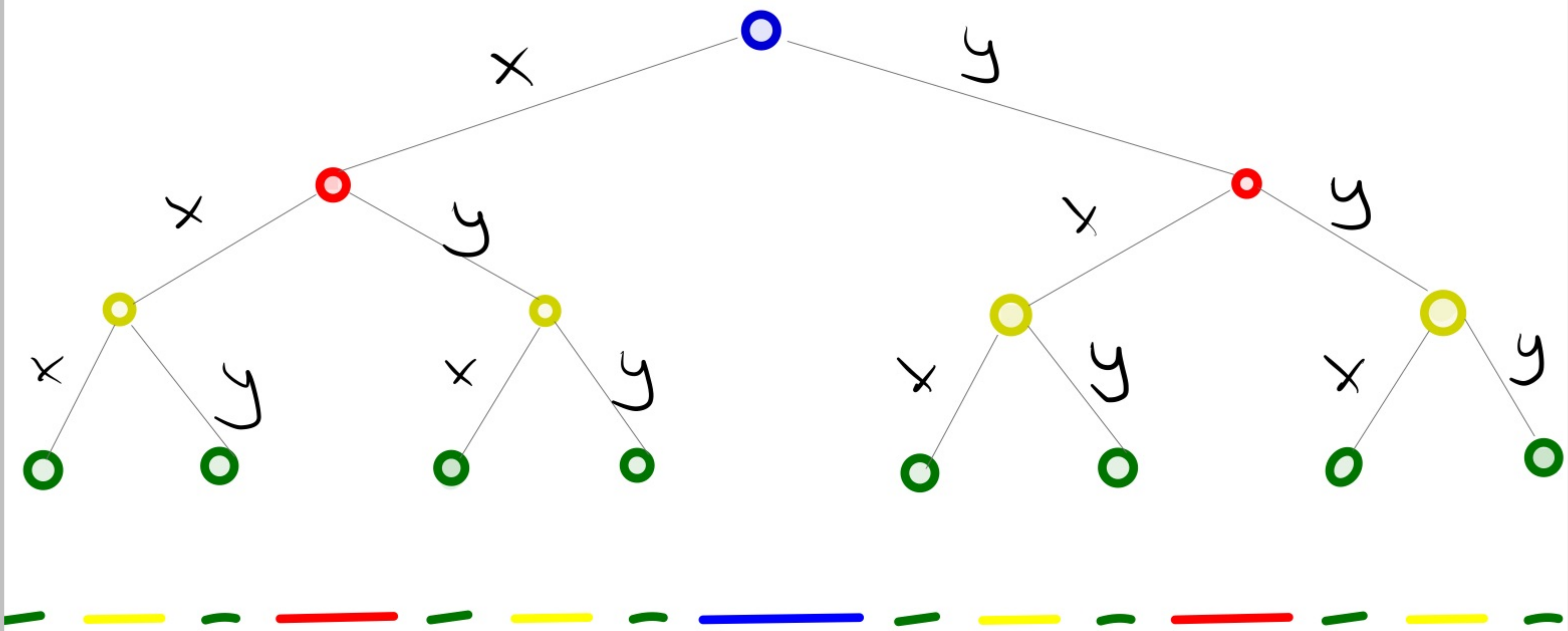


Tree of Mutations



We can mutate from the x, v, or y corner





x
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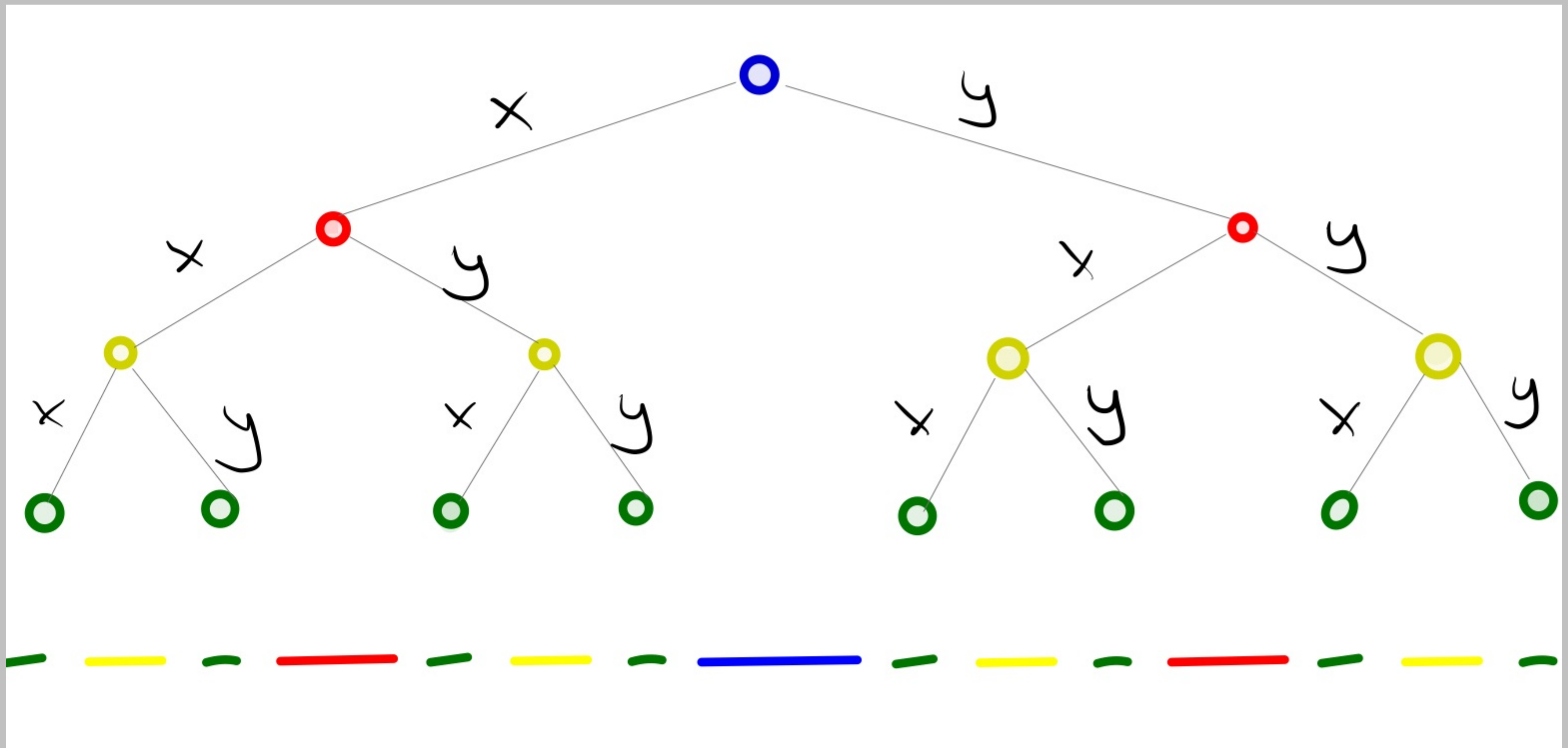
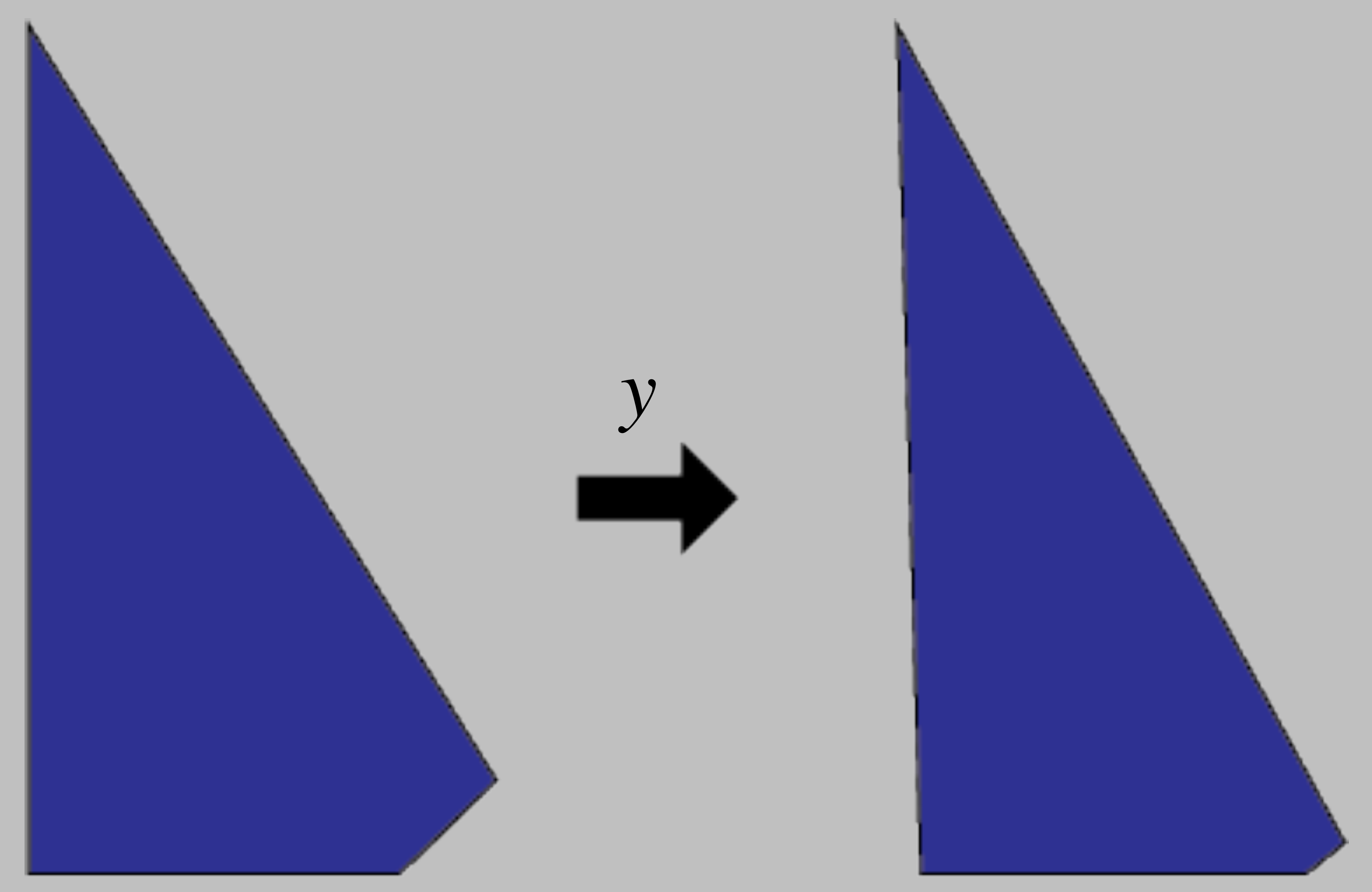


For an interval I in $(\frac{n}{n+1}, \frac{n+1}{n+2})$ to construct the embedding for the left endpoint of I,

- Perform $n+2$ mutations by v.
- Perform the mutations to get to the vertex in the graph corresponding to I.

Then, perform consecutive mutations by y.

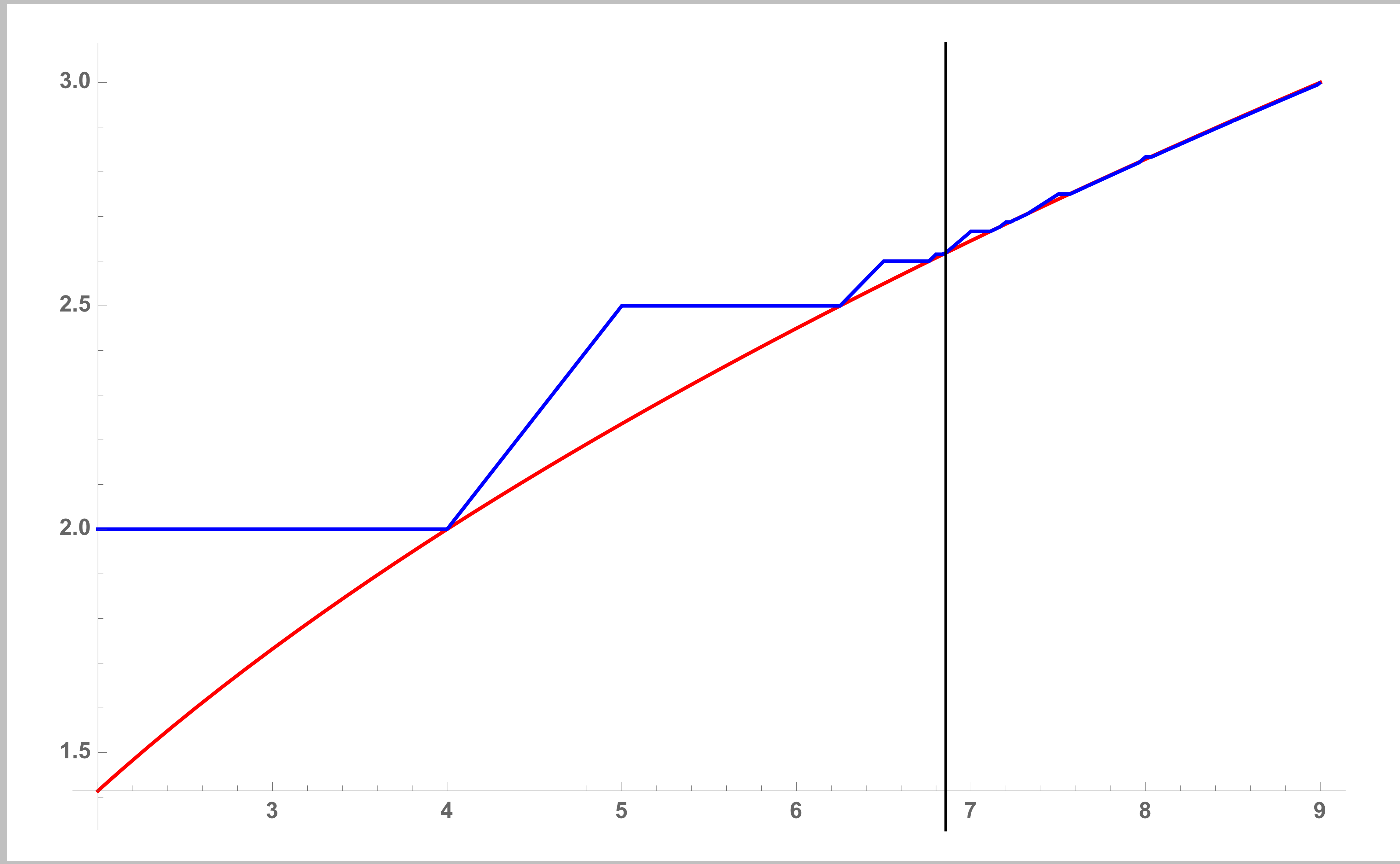
For the proof, the numerics for the classes completely determine the numerics of the different fibrations.



ATFs expected to compute all embeddings for small z

The sequences of mutations we considered for each specific b only construct an optimal embedding at accumulation point.

Preliminary evidence suggests that for many b different sequence of mutations will compute all embeddings on the embedding function before the accumulation point (i.e. for increasing staircases)



Main Result

M., McDuff, Weiler

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- For each n, Block $\cap \left(\frac{n}{n+1}, \frac{n+1}{n+2}\right)$ is homeomorphic to the complement of the middle third Cantor set.
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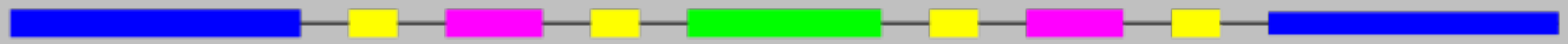
$\frac{1}{2} < b$



$b < \frac{2}{3}$