Approximation of Generating Function Barcode for Hamiltonian Diffeomorphisms

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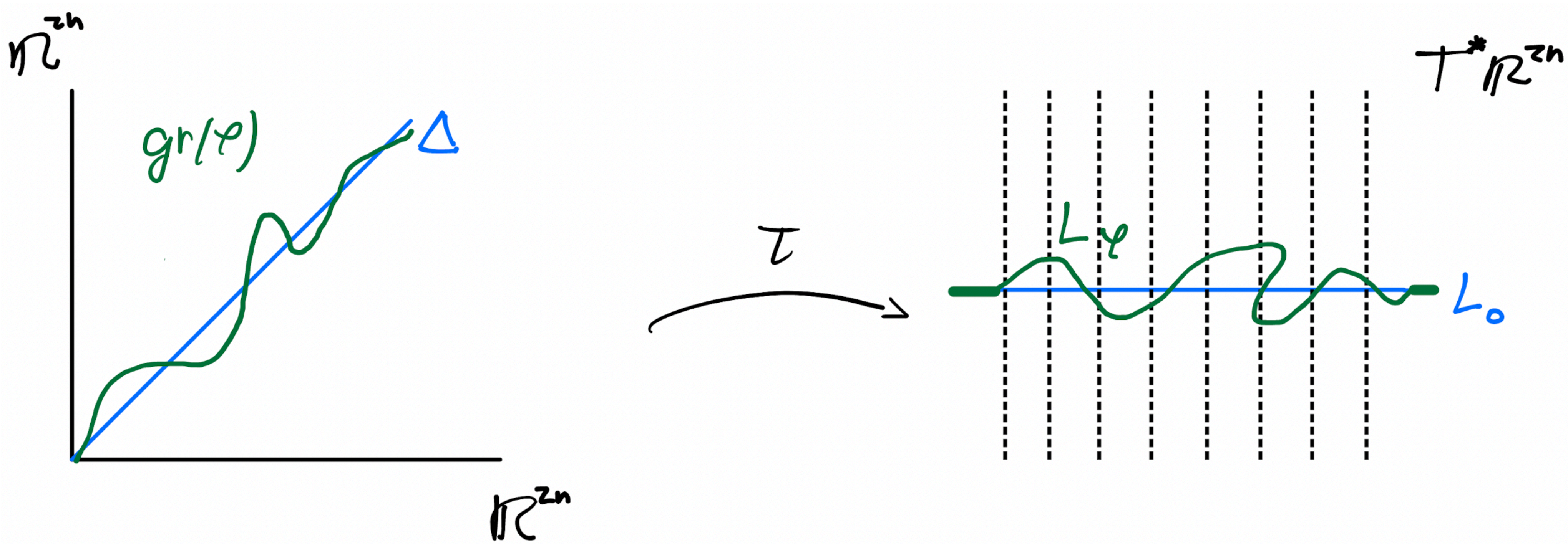
- Symplectic Zoominar -

Joint work with Pazit Haim-Kislev

Goal: define a conjugation invariant *barcode* of compactly supported Hamiltonian diffeo' $\varphi = \varphi_N \circ \cdots \circ \varphi_1$ of $\mathbb{R}^{2n}$, that can be numerically approximated from samples of generating functions of $\varphi_1, \ldots, \varphi_N$.

Generating functions

$$
\begin{array}{ccccccc}
\mathrm{Ham}_c(\mathbb{R}^{2n}) & & \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} & & T^*\mathbb{R}^{2n} & & T^*\mathbb{S}^{2n} \\
\Cup & & \cup & & \cup & & \cup \\
\varphi & \xrightarrow{\text{graph}} & \mathrm{gr}(\varphi) & \xrightarrow{\text{Lagrangian}} & L_\varphi & \xrightarrow{\text{compactify}} & \overline{L_\varphi}
\end{array}
$$



if φ is C^1-close to Id, then $L_\varphi = \mathrm{gr}\,(dS)$, for $S : \mathbb{R}^{2n} \to \mathbb{R}$

Definition: $S\colon \mathbb{R}^{2n}_x \times \mathbb{R}^d_\xi \to \mathbb{R}$ is a generating function of $L \subset T^*\mathbb{R}^{2n}$ if

$$L = \left\{ \left(x, \frac{\partial S}{\partial x} \right) \in T^*\mathbb{R}^{2n};\ \frac{\partial S}{\partial \xi} = 0 \right\}, \qquad \frac{\partial S}{\partial \xi} \pitchfork 0$$

Generating function homology

$$\left\{\begin{array}{c}\text{(non-deg') fixed}\\ \text{points of } \varphi\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{(transversal) intersection}\\ \text{points of } L_\varphi \cap L_0\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{(non-deg') crit'}\\ \text{points of } S\end{array}\right\}$$

- S is a GFQI if $S(x,\xi) = \mathcal{Q}(\xi)$ (non-degenerate quadratic form) outside a compact set B. Extend to $S : \mathbb{S}^{2n} \times \mathbb{R}^d \to \mathbb{R}$
- for $L_0 \overset{\text{Ham'}}{\sim} L \subset T^*\mathbb{S}^{2n}$, all GFQI are equivalent (Viterbo)
- for $L_0 \overset{\text{Ham'}}{\sim} L \subset T^*\mathbb{S}^{2n}$, there exists a GFQI (Laudenbach, Sikorav)

Definition: The GF homology of φ with GFQI S, w.r.t $t \notin \text{Crit}(S)$ is

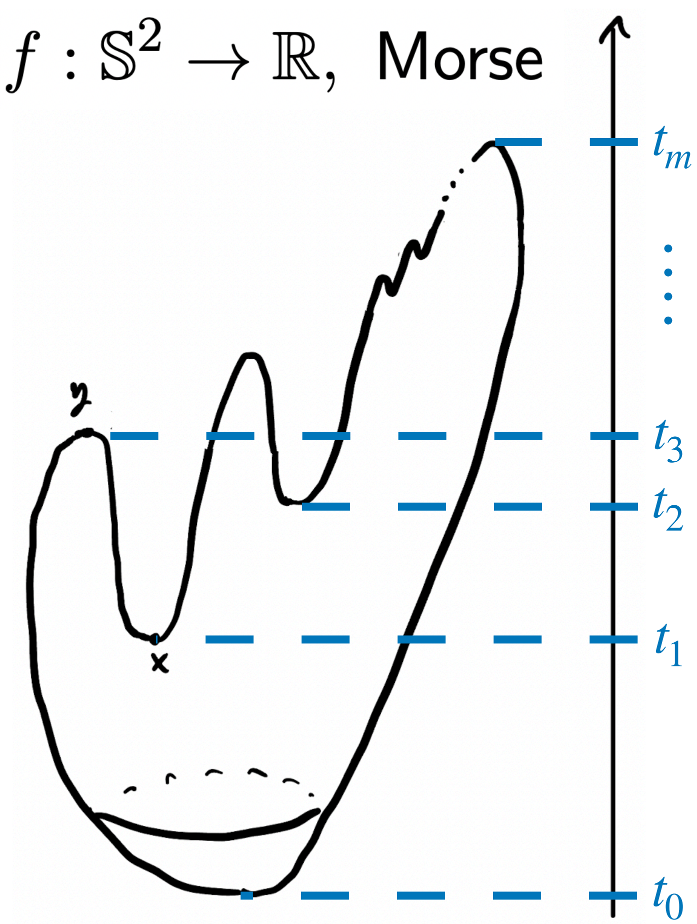
$$G_*^{(-\infty,t]}(\varphi) := H_{*+i}(\{S \le t\}, \{S \le a\}), \quad a < \min S|_B,\ i := \text{ind}(\mathcal{Q}).$$

- $G_*^{(-\infty,t]}(\varphi)$ are independent of the choice of S and invariant under conjugation by $\psi \in \text{Symp}(\mathbb{R}^{2n})$ (Traynor)

Generating function barcode

- a barcode is a finite collection of intervals $I_j = (a_j, b_j]$ such that $-\infty < a_j < b_j \leq \infty$, with multiplicities $m_j \in \mathbb{N}$
- metric: $d_{\text{bot}} = \inf \delta > 0$ s.t after removing short bars ($< 2\delta$), we match the others by aligning their endpoints with distance $< \delta$.

$f : \mathbb{S}^2 \to \mathbb{R}$, Morse



$$V_m = H_*(f < t_m + \varepsilon)$$
$$\pi^{m-1,m} \uparrow$$
$$\vdots$$
$$\pi^{1,2} \uparrow$$
$$V_1 = H_*(f < t_1 + \varepsilon)$$
$$\pi^{0,1} \uparrow$$
$$V_0 = H_*(f < t_0 + \varepsilon)$$

$\pi^{i,j} : V_i \to V_j$, $(i \leq j)$
a class $\alpha \in V_i$
is born at t_i if:
$\alpha \notin \mathrm{Im}\left(\pi^{i-1,i}\right)$
and dies at t_j if:
$\pi^{i,j-1}(\alpha) \notin \mathrm{Im}\left(\pi^{i-1,j-1}\right)$
and $\pi^{i,j}(\alpha) \in \mathrm{Im}\left(\pi^{i-1,j}\right)$

- bars $(t_i, t_j]$ for classes of critical pts that are born at t_i and die at t_j
- rays (t_i, ∞) for classes of critical pts that are born at t_i and never die

Generating function barcode

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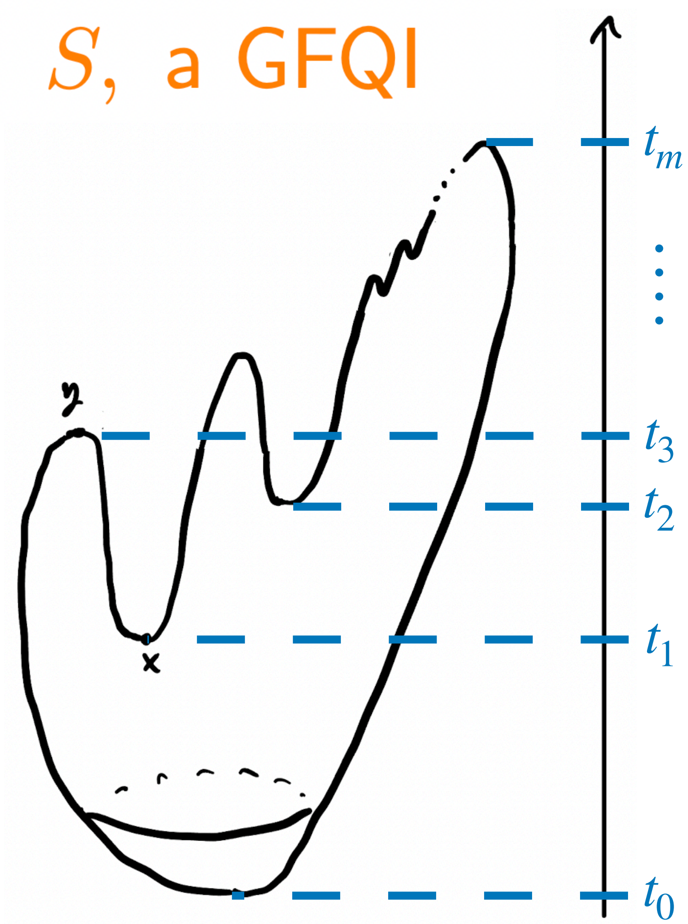

$$V_m = G_*^{(-\infty, t_m + \varepsilon]}(\varphi)$$
$$\uparrow \pi^{m-1,m}$$
$$\vdots$$
$$\uparrow \pi^{1,2}$$
$$V_1 = G_*^{(-\infty, t_1 + \varepsilon]}(\varphi)$$
$$\uparrow \pi^{0,1}$$
$$V_0 = G_*^{(-\infty, t_0 + \varepsilon]}(\varphi)$$

Definition: The GF barcode $\mathcal{B}(\varphi)$ is the barcode associated to $G_*^{(-\infty, t]}(\varphi)$

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Approximation algorithm

- $S_1, \ldots, S_N : \mathbb{R}^{2n} \to \mathbb{R}$, generating $\varphi_1, \ldots, \varphi_N \in \mathrm{Ham}_c\left(\mathbb{R}^{2n}\right)$, such that $\mathrm{supp}\left(\varphi_j\right) \subset \mathbb{B}_R$ and $\|\varphi_j - \mathrm{Id}\|_{C^1} < T$

$\rightsquigarrow$ composition formula (Chekanov, Chaperon)

- $S : \mathbb{R}^{2nN} \to \mathbb{R}$ a GFQI of $\varphi = \varphi_N \circ \cdots \circ \varphi_1$, equal to $\mathcal{Q}$ outside B
- construct a pair (K, L) of finite CW complexes with mesh $\frac{1}{m}$, homotopic to $\left(\mathbb{S}^{2n} \times \mathbb{R}^d, \{S \leq a\}\right)$ with $a < \min S|_B$

$\rightsquigarrow$ sample $S|_{B \cap \frac{1}{m}\mathbb{Z}^{2nN}}$ to get a filtered complex $\left(K^t, L\right)$

- boundary matrices ∂_j with rows and columns ordered by the filtration

$\rightsquigarrow$ matrix reduction (Edelsbrunner, Letscher, Zomorodian)

- approximated barcode $\mathcal{B}$

Theorem: $d_{\text{bot}}\left(\mathcal{B}, \mathcal{B}(\varphi)\right) \leq C(R) \cdot \frac{\sqrt{n} T N^2}{m} =: \varepsilon$,
polynomial time complexity in m, super-exponential in N (for fixed ε)

Computational experiment

- A MATLAB program was implemented with input:

$$\underbrace{\{0 < R_j\}_{j=1}^N}_{\text{support radii}} \quad \underbrace{\{c_j\}_{j=1}^N \subset \mathbb{R}^2}_{\text{center points}} \quad \underbrace{\{h_j \in C^\infty([0, R_j^2/2])\}_{j=1}^N}_{\text{profile functions}}$$

and parameters $T, T' > 0, m \in \mathbb{N}$ such that $\left|h_j'\right| \leq T, \left|h_j''\right| \leq T'$.

- approximates samples of $\{S_j : \mathbb{R}^2 \to \mathbb{R}\}_{j=1}^N$ generating $\varphi_{H_j}^1$ where $H_j(x) = h_j\left(\frac{|x-c_j|^2}{2}\right)$, using the Hamilton-Jacobi equation.

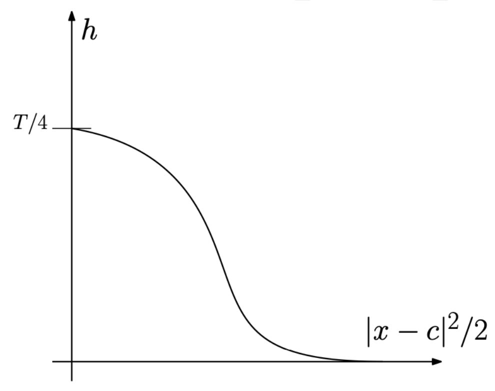
- returns a barcode $\mathcal{B}$ such that

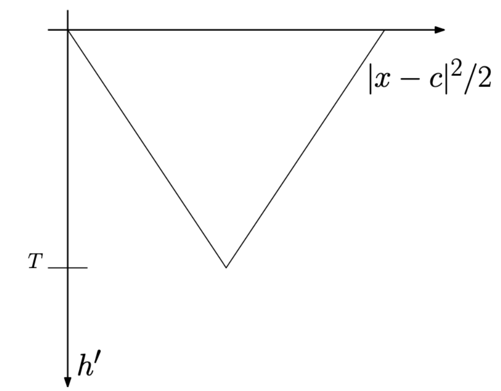
$$d_{\text{bot}}(\mathcal{B}, \mathcal{B}(\varphi)) \leq C(R)\frac{TN^2}{m} + C_1(T, T', R)\frac{N}{m} + C_2(T, T', R) \cdot N\sqrt{E},$$

where E is a numerical error term

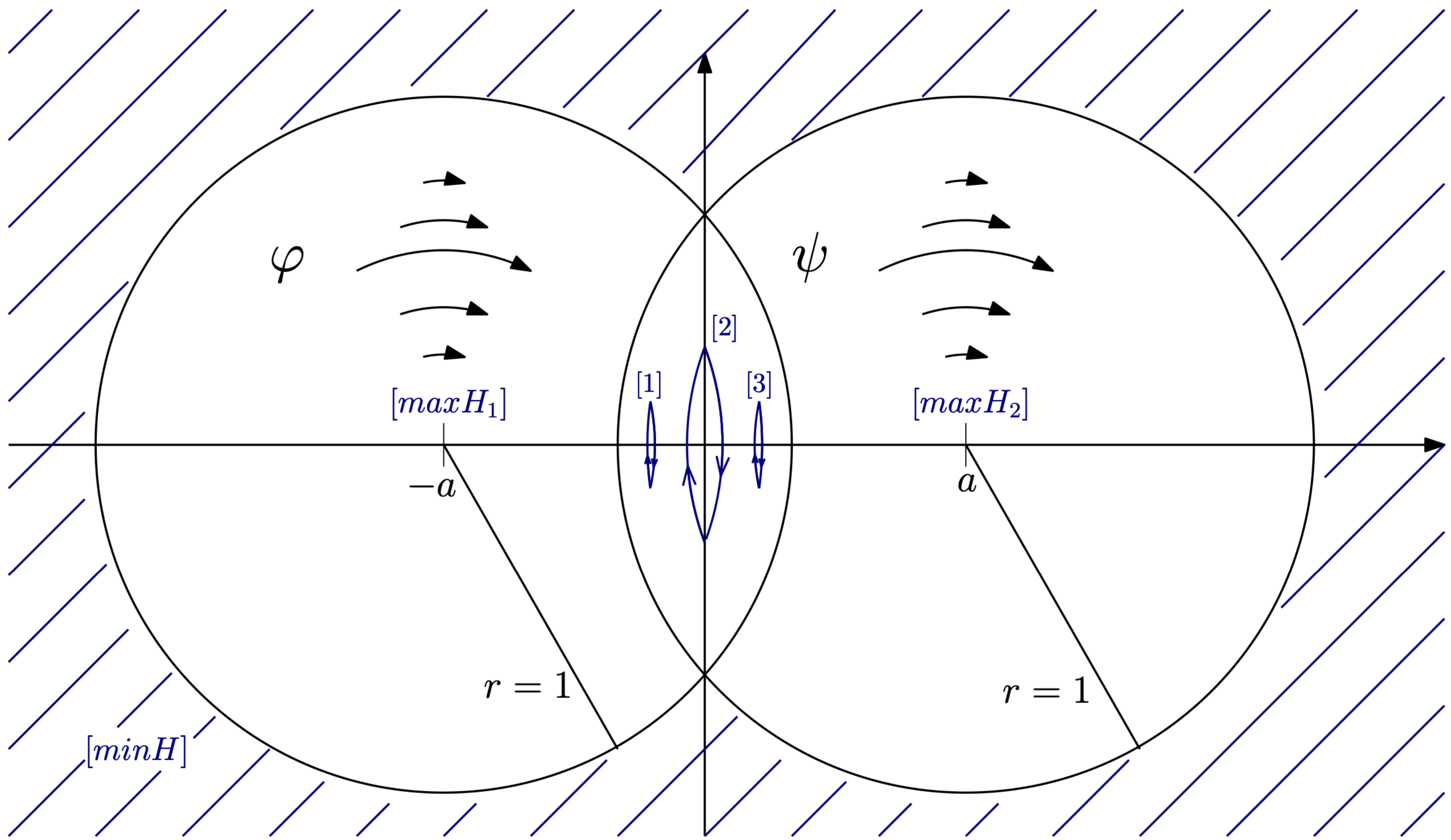
Computational experiment

- Consider the profile $h : \left[0, \frac{1}{2}\right] \to \mathbb{R}$ given by

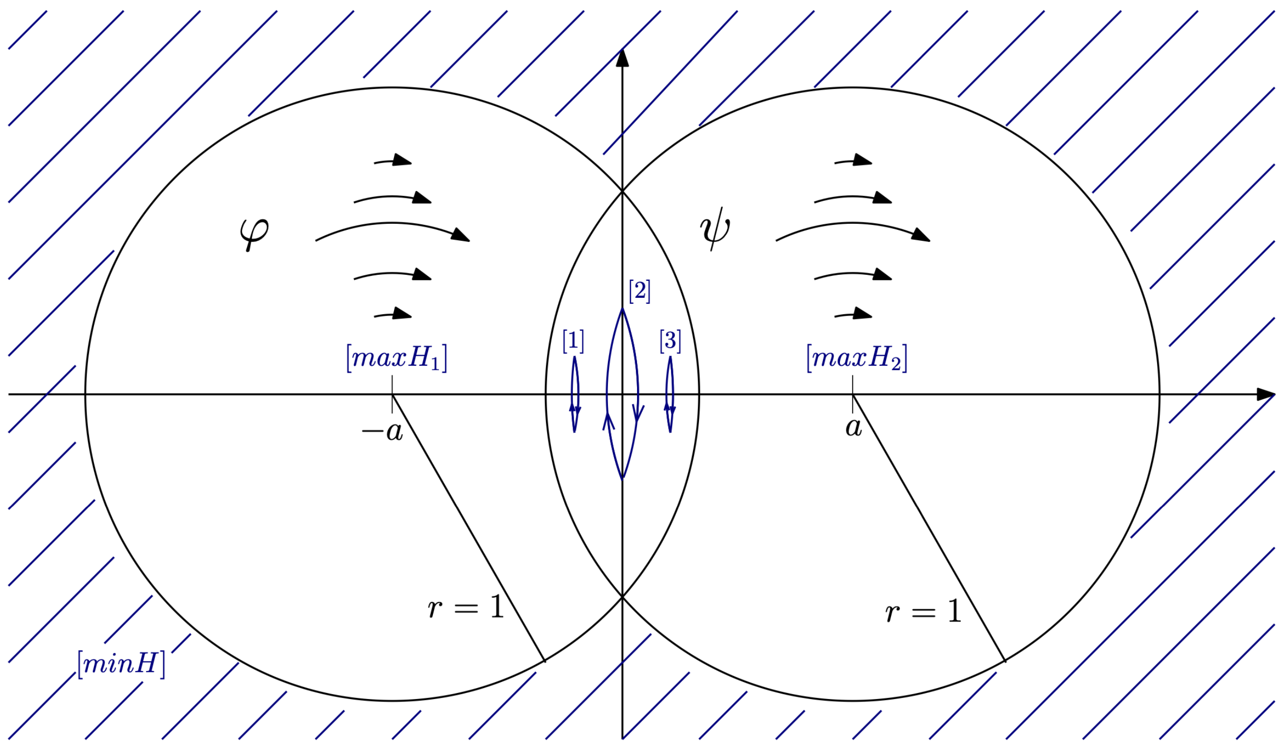




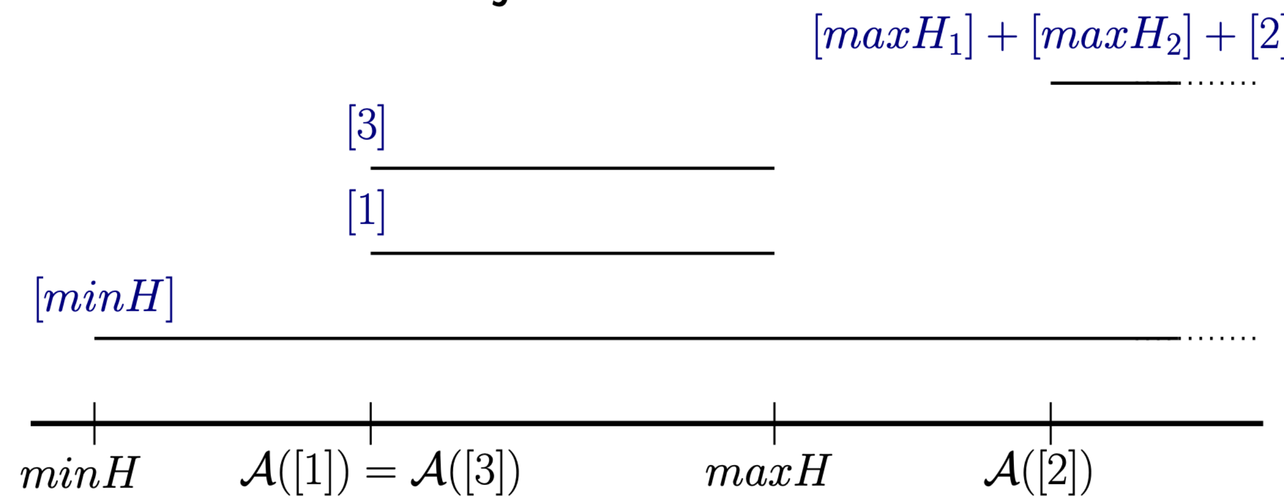
- for $a > 0$, denote center points $c_1 = (-a, 0)\,, c_2 = (a, 0) \in \mathbb{R}^2$



Computational experiment



conjectured barcode



computation results

a	Estimated longest finite bar$/T$	$d_{\text{bot}}(\mathcal{B}(\psi\circ\varphi),\mathcal{B})/T$
0.70	$\sim 1.57 \times 10^{-2}$	$\sim 2.03 \times 10^{-6}$

Thank you!