A microlocal invitation to Lagrangian fillings

Symplectic Zoominar – CRM-Montréal, Princeton/IAS, Tel Aviv, and Paris

Roger Casals (UC Davis)
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Legendrian links

**Contact topology:** studying Legendrian submanifolds is useful

Legendrian front
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- Detection of Reeb orbits, computation of Floer-theoretic invariants, classification of contact structures, connections to other areas. They also appear in nature and are beautiful in their own right.
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- Detection of Reeb orbits, computation of Floer-theoretic invariants, classification of contact structures, connections to other areas. They also appear in nature and are beautiful in their own right.
- Today we consider **Legendrian links** \( \Lambda \subset (T^*_\infty \mathbb{R}^2, \xi_{st}) \).
**A microlocal start**

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(i) **Toy example**: For subsets $A, B \subset \mathbb{R}^2$ with characteristics $\chi_A, \chi_B : \mathbb{R}^2 \rightarrow \{0, 1\}$, intersection $A \cap B$ captured by product $\chi_A \cdot \chi_B$. 
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**(ii) Idea**: Since every Legendrian link in \( T^*_\infty \mathbb{R}^2 \) has a front \( \pi(\Lambda) \subset \mathbb{R}^2 \), study constructible functions with respect to the stratification \( \pi(\Lambda) \).
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(ii) **Idea**: Since every Legendrian link in $T^*_\infty \mathbb{R}^2$ has a front $\pi(\Lambda) \subset \mathbb{R}^2$, study constructible functions with respect to the stratification $\pi(\Lambda)$.

(iii) The right setup: study constructible *sheaves*. The notion of “first derivative” is captured by the *singular support*, pioneered by Mikio Sato.
Categories of sheaves on $\mathbb{R}^2$ singularly supported on a front

The category: For a Legendrian link in $T^*_\infty \mathbb{R}^2$. Consider the dg-derived category $\mathcal{C}(\Lambda)$ of constructible sheaves on $\mathbb{R}^2$ with singular support on $\Lambda$. In particular, constructible with respect to the front $\pi(\Lambda) \subset \mathbb{R}^2$. 
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(ii) There is a **geometric moduli of objects** $\mathcal{M}(\Lambda)$ for $\mathcal{C}(\Lambda)$ by Toën-Vaquié.

(iii) $\mathcal{M}(\Lambda) = \{(v_1, v_2, v_3, v_4, v_5) : v_i \in \mathbb{C}^2, \det(v_i, v_{i+1}) = 1, i \in \mathbb{Z}_5\} / \text{PGL}_2(\mathbb{C})$

Set $v_1 = (1, 0), v_2 = (0, 1), v_3 = (1, z_1), v_4 = (z_4, z_3), v_5 = (z_2, -1)$.

Then $\mathcal{M}(\Lambda) = \{z_3 + z_1 + z_1z_3z_2 = 1\} \subset \mathbb{C}^3_{z_1, z_2, z_3}$. 
Theorem (Main Theorem)

Existence and explicit construction of **quasi-cluster A-structures** on moduli \( \mathcal{M}(\Lambda) \) of sheaves with singular support on \( \Lambda \), for many Legendrians \( \Lambda \subset (\mathbb{R}^3, \xi_{st}) \). In particular, \( \mathbb{C}[\mathcal{M}(\Lambda)] \) is a **cluster algebra**.

Legendrian front
Simplified Main Result

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(iii) Why is it useful to have cluster $A$-structures? \(\rightarrow\) Solves several open problems.
The Main Result gives a fruitful bridge

Use results from cluster algebras to prove results in symplectic topology:

**Corollary (with H. Gao, Annals'22)**

"Infinitely many Lagrangian fillings."

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**Corollary (Leclerc’s Conjecture)**

Let $u, v \in S_n$ be two permutations, $u \leq v$, and $R(u, v)$ their Richardson variety. Then $\mathbb{C}[R(u, v)]$ is a cluster algebra.

This latter result is a combination of work with D. Weng and separately with E. Gorsky et al. The proof is relatively simple: construct a $\Lambda = \Lambda_{u,v}$ such that $\mathcal{M}(\Lambda_{u,v}) \cong R(u, v)$ and apply main result.
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Both corollaries are actually stronger, including other Lie types. This has opened fertile ground for more (→ e.g. AIM Workshop on Jan’23.)
Moduli of Lagrangian Fillings

Symplectic Geometry: Study Lagrangian fillings of Legendrian links
1. Consider a **Legendrian link** $\Lambda \subset (T^*\mathbb{R}^2, \xi_{st}) \cong (\mathbb{R}^2 \times S^1_\theta, \ker(d\theta - ydx))$. 

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- Lagrangian filling gives $(\mathbb{C}^*)^{b_1(L)} \subset \mathcal{M}(\Lambda)$ chart.
  (Lagr. filling with Abelian local system gives point in $\mathcal{M}(\Lambda)$.)
The intuition for cluster varieties

**Definition**

A *cluster $A$-variety* $\mathcal{M}$ is a union $\mathcal{M} \overset{(cd.2)}{=} \bigcup_{s \in S} T_s$, $T_s \cong (\mathbb{C}^*)^d$ algebraic tori, with a given identification $\text{Spec } T_s \cong \mathbb{C}[A_{s,1}^{\pm 1}, \ldots, A_{s,d}^{\pm 1}]$ such that, in these identifications, the transition functions are $A$-mutations $\mu_{A_s,i}$. For us a Lagrangian filling gives toric chart, but what does symplectically gives the coordinates $A_s$, $j$ and these transition functions?
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- **Outstanding geometry**: computation of **singular cohomology**, with mixed Hodge structure, existence of **holomorphic symplectic** form, with curious Lefschetz, $\mathbb{F}_q$-**point counts**, any more. (E.g. $H^*(\mathcal{M}(\Lambda_{819}), \mathbb{C})$.)
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- Trefoil Example: Then $\mathcal{M}(\Lambda_{3_1}) = \{z_1 + z_3 + z_1z_2z_3 + 1 = 0\} \subset \mathbb{C}^3$, quiver is $\bullet \rightarrow \bullet$ and we have five algebraic tori:
  
  $T_1 = \text{Spec}\{z_1^{\pm 1}, (1+z_1z_2)^{\pm 1}\}$,  
  $T_2 = \text{Spec}\{z_3^{\pm 1}, (1+z_3z_2)^{\pm 1}\}$,  
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Lagrangian Disk Surgeries

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The first symplectic fact towards cluster algebras: **Lagrangian surgery**.

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(iii) How do you find these? \( \rightarrow \) **Legendrian weaves** \( (G&T \ '22, \ 116p)\). See also “Microlocal Theory of Legendrian Links and Cluster Algebras” \( (119p)\).
Summary Thus Far

The key points at this stage

Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{st}) \mapsto D^-$-stack $M(\Lambda)$ of objects in $\mathcal{Sh}_\Lambda(\mathbb{R}^2)$.

(i) $M(\Lambda)$ acts as "space of Lagrangian fillings", in that an embedded exact Lagrangian $L \subset (\mathbb{R}^4, \lambda_{st})$, $\partial L = \Lambda$, with local system, gives a point in $M(\Lambda)$. Focus on Abelian local systems $H_1(L, \mathbb{C}^*)$, then:

Lagrangian filling $L \mapsto (\mathbb{C}^*)_{b_1(L)} \subset M(\Lambda)$ toric chart.

(ii) Given $L$-compressible cycle $\gamma \subset L$, $\gamma$-surgery gives new filling $\mu_\gamma(L)$, and thus new toric chart in $M(\Lambda)$. Need regular functions from $L$.

(iii) Need $\Lambda$ such that $D^-$-stack $M(\Lambda)$ is accessible, e.g. affine variety or algebraic quotient thereof, so cluster structures make sense: $\mapsto$ Legendrian links $\Lambda$ from grid plabic graph $G$ or $(-1)$-closures of braids.
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$\leadsto$ Legendrian links $\Lambda$ from grid plabic graph $\mathbb{G}$ or (-1)-closures of braids.
The picture to take home

$(T^* \mathbb{R}^2, \omega_{st})$

Lagrangian skeleton

$\mathbb{L}$-compressible

$\gamma_1$

$\gamma_2$

$\gamma_3$

$L$-local system

$A_{\gamma_1} : \mathcal{M}(\Lambda) \rightarrow \mathbb{C}$

Microlocal holonomy

$A_{\gamma_1} : (\mathbb{C}^*)^b_1(\mathbb{L}) \rightarrow \mathbb{C}$
Build $\mathbb{L}$-incompressible system $\rightarrow$ relative Lagrangian skeleton of $(\mathbb{C}^2, \Lambda)$.
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The special coordinates $A_\gamma$ are **microlocal holonomies** along dual relative cycles. **Miracle**: they are regular on $\mathcal{M}(\Lambda)$!
Legendrian links $\Lambda(G)$ & Grid Plabic Graphs $G$

By definition, a **grid plabic graph** $G \subset \mathbb{R}^2$ is:

![Diagram of a grid plabic graph]

The alternating strand diagram associated to $G$ is drawn as follows:

Then, $\Lambda(G) \subset (\mathbb{R}^3, \xi_{st})$ is the Legendrian link associated to this front, after satelliting the Legendrian $S^1$-fiber of $T^*\mathbb{R}^2$ to the standard unknot.
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Examples of $\mathcal{M}(\Lambda(G))$

**Positive braids:** $G$ plabic fence for $\beta = \sigma_{i_1} \ldots \sigma_{i_s} \in \Br_n^+$. Then $\mathcal{M}(\Lambda(G))$ is the moduli of tuples of affine flags in $(GL_n/U)^{s+n(n-1)}$ with $F_j, F_{j+1}$ in $s_{ij}$-relative position, with a $\Delta_n^2$, plus framing conditions. ([CGGS 1&2])

E.g., for $[\beta] = T(k, n)$, $\mathcal{M}(\Lambda(G)) \cong \Gr(k, n+k) \setminus \{\Delta_{1,2} \cdots \Delta_{n+k,1} = 0\}$.
**Examples of $\mathcal{M}(\Lambda(\mathcal{G}))$**

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**Example $m(5_2)$:** $\mathcal{M}(\Lambda(\mathbb{G}))$ involves incidences of flags in varying $\mathbb{P}^k$'s.

Some degenerations allowed, but some not!
The key points for these Legendrians

**Theorem A:** Let $\mathcal{G}$ be a GP-graph. Then

$$\exists \vartheta(\mathcal{G}) \text{ weave } \leadsto \text{ embedded Lagrangian filling } L(\mathcal{G}) + \text{ basis of } Y\text{-cycles}$$

Plus, we can read $\mathbb{L}$-compressible l.i. cycles from $\mathcal{G}$ combinatorially.
The key points for these Legendrians

- **Theorem A**: Let $G$ be a GP-graph. Then

  $$\exists \mathfrak{w}(G) \text{ weave } \leadsto \text{ embedded Lagrangian filling } L(G) + \text{ basis of } Y\text{-cycles}$$

  Plus, we can read $L$-compressible l.i. cycles from $G$ combinatorially.

- **Theorem B**: $M(\Lambda(G))$ is isomorphic to the moduli of solutions of an incidence problem of affine flags in varying $C^k$'s such that

  $$\mathfrak{w}(G) \text{ weave } \leadsto T_{\mathfrak{w}(G)} \subset M(\Lambda(G)) \text{ open toric chart}$$

  Moreover, $T_{\mathfrak{w}(G)} \cong (\mathbb{C}^*)^d$ from further flag transversality conditions.
The key points for these Legendrians

- **Theorem A:** Let $\mathcal{G}$ be a GP-graph. Then
  \[ \exists \varpi(\mathcal{G}) \text{ weave } \overset{s.t.}{\rightsquigarrow} \text{ embedded Lagrangian filling } \mathcal{L}(\mathcal{G}) + \text{ basis of } \mathcal{Y}-\text{cycles} \]

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- **Theorem B:** $\mathcal{M}(\Lambda(\mathcal{G}))$ is isomorphic to the moduli of solutions of an incidence problem of affine flags in varying $\mathbb{C}^k$'s such that
  \[ \varpi(\mathcal{G}) \text{ weave } \overset{\text{gives}}{\sim} T_{\varpi(\mathcal{G})} \subset \mathcal{M}(\Lambda(\mathcal{G})) \text{ open toric chart} \]

  Moreover, $T_{\varpi(\mathcal{G})} \cong (\mathbb{C}^*)^d$ from further flag transversality conditions.

- **Next: Theorem C.** Need to introduce the basis of regular functions:
  \[ \varpi(\mathcal{G}) \text{ weave } \overset{\text{gives}}{\sim} T_{\varpi(\mathcal{G})} \text{ open toric chart } + \text{ basis of } \mathbb{C} [T_{\varpi(\mathcal{G})}] \]

  In addition, this basis $\mathbb{C} [T_{\varpi(\mathcal{G})}]$ must change according to cluster $A$-mutation for $Q(B(\mathcal{G}))$ when Lagrangian surgery is performed.
Define candidate \textit{A-variables} with Guillermou-Kashiwara-Schapira maps:

\[ \mathbb{I} \text{Sh}_\Lambda(\mathbb{R}^2) \longrightarrow \mu \text{Sh}_\Lambda, \quad \mu \text{Sh}_\Lambda(\Lambda) \cong \text{Loc}(\Lambda), \]

where \( \Lambda \) is a Legendrian. This is used twice: \( \Lambda = \tilde{\Lambda}(G) \) and \( \Lambda = \Lambda(G) \).
The microlocal local system on $L(G)$ and $\Lambda(G)$

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where $\Lambda$ is a Legendrian. This is used twice: $\Lambda = \tilde{L}(G)$ and $\Lambda = \Lambda(G)$.

(1) **Upshot:** Each point in $\mathcal{M}(G)$ defines a local system in $\Lambda(G)$, and each point in the $\mathfrak{w}(G)$ toric chart defines a local system in $L(G)$. 
The microlocal local system on $L(\mathcal{G})$ and $\Lambda(\mathcal{G})$

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1) **Upshot**: Each point in $\mathcal{M}(\mathcal{G})$ defines a local system in $\Lambda(\mathcal{G})$, and each point in the $\mathfrak{w}(\mathcal{G})$ toric chart defines a local system in $L(\mathcal{G})$.

2) **Theorem**: This parallel transport can be computed by using cones in the braid slice of a weave: *ratios of wedges of decorations.*
Microlocal Merodromies

Definition (Key new concept)

Let $\mathcal{G}$ be a GP-graph and $B(\mathcal{G})$ the dual relative basis of Y-cycles of the weave $\varpi(\mathcal{G})$. The microlocal merodromy along $\eta \in B(\mathcal{G})$ is

$$A_\eta : M(\mathcal{G}) \rightarrow \mathbb{C}$$

where $A_\eta(F^\bullet) =$ “transport decorations of $F^\bullet$ in $\partial \eta$ and compare”.

Theorem (The Technical Properties)

The set of microlocal merodromies $\{A_\eta\}$ satisfies:

1. $\mu_\gamma(A_\eta)$ is a cluster $A$-mutation on $A_\eta$ if $\gamma$ absolute Y-tree dual to $\eta$.
2. $A_\eta$ and adjacent $\mu_\gamma(A_\eta)$ are irreducible and regular functions.
3. $A_f$ is a unit if and only if non-sugar free hull.

These properties are not true unless $\eta \in B(\mathcal{G})$!
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These properties are not true unless $\eta$ belongs to $B(G)$!
Finally, after developing these results, we can conclude:

**Theorem (Simplified Upshot)**

The moduli $\mathcal{M}(G)$ admits a **cluster $\mathcal{A}$-structure** in its coordinate ring, with initial cluster seed as **symplectically described**.

The crucial step is showing that the inclusion of the upper bound into $\mathcal{M}(G)$ is an isomorphism, up to codimension 2. This is done by applying "Technical Properties" and an argument with immersed weaves.
The resulting cluster $A$-structure

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- The stronger theorem being proved is in great part symplectic geometric: ability to define cluster $A$-coordinate symplectically via merodromies on Lagrangian fillings and a basis of dually $L$-compressible relative cycles.
Thanks a lot!

"But this is the simplified version for the general public."