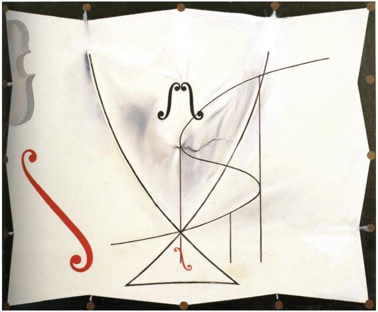
A microlocal invitation to Lagrangian fillings

Symplectic Zoominar – CRM-Montréal, Princeton/IAS, Tel Aviv, and Paris



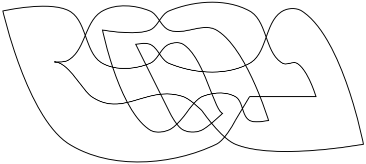
Roger Casals (UC Davis)

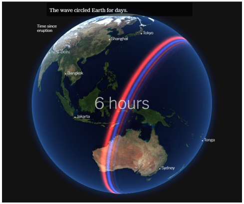
November 11th 2022

Legendrian links

Contact topology: studying Legendrian submanifolds is useful

Legendrian front





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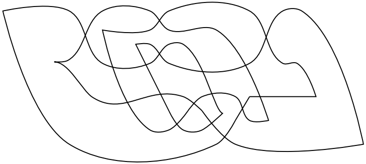

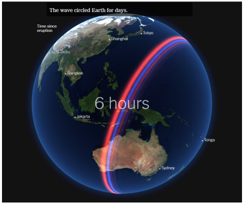

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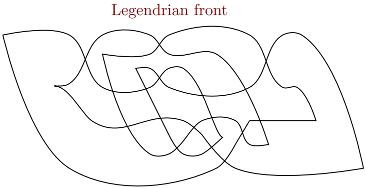
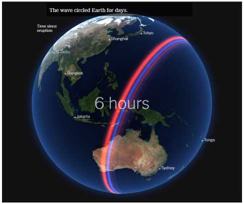
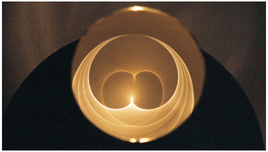
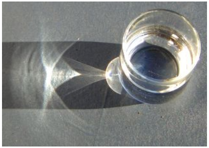




- Detection of Reeb orbits, computation of Floer-theoretic invariants, classification of contact structures, connections to other areas. They also appear in nature and are beautiful in their own right.
- Today we consider **Legendrian links** $\Lambda \subset (T^*_\infty \mathbb{R}^2, \xi_{st})$.





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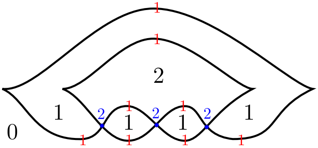
(ii) *Idea*: Since every Legendrian link in $T^*_\infty \mathbb{R}^2$ has a front $\pi(\Lambda) \subset \mathbb{R}^2$, study constructible functions with respect to the stratification $\pi(\Lambda)$.



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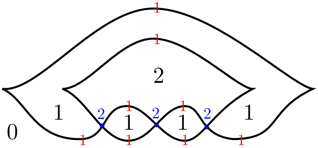

(iii) The right setup: study construcible *sheaves*. The notion of "first derivative" is captured by the *singular support*, pioneered by Mikio Sato.

Categories of sheaves on $\mathbb{R}^2$ singularly supported on a front

The category: For a Legendrian link in $T^*_\infty \mathbb{R}^2$. Consider the dg-derived category $\mathcal{C}(\Lambda)$ of constructible sheaves on $\mathbb{R}^2$ with singular support on Λ.* In particular, constructible with respect to the front $\pi(\Lambda) \subset \mathbb{R}^2$.

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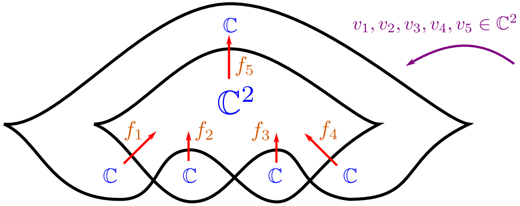
(iii) $\mathfrak{M}(\Lambda) = \{(v_1, v_2, v_3, v_4, v_5) : v_i \in \mathbb{C}^2, \det(v_i, v_{i+1}) = 1, i \in \mathbb{Z}_5\}/PGL_2(\mathbb{C})$



$v_1 = im(f_1)$
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$v_5 = ker(f_5)$

Set $v_1 = (1, 0), v_2 = (0, 1), v_3 = (1, z_1), v_4 = (z_4, z_3), v_5 = (z_2, -1)$.
Then $\mathfrak{M}(\Lambda) = \{z_3 + z_1 + z_1 z_3 z_2 = 1\} \subset \mathbb{C}^3_{z_1,z_2,z_3}$.

Simplified Main Result

Theorem (Main Theorem)

Existence and explicit construction of **quasi-cluster $\mathcal{A}$-structures** *on moduli $\mathfrak{M}(\Lambda)$ of sheaves with singular support on Λ, for many Legendrians $\Lambda \subset (\mathbb{R}^3, \xi_{st})$. In particular, $\mathbb{C}[\mathfrak{M}(\Lambda)]$ is a cluster algebra.*

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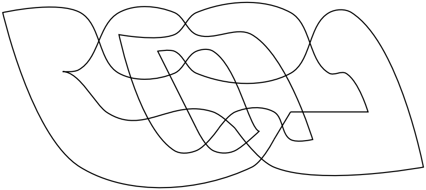

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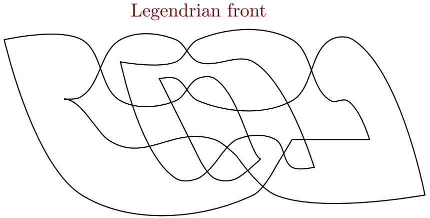

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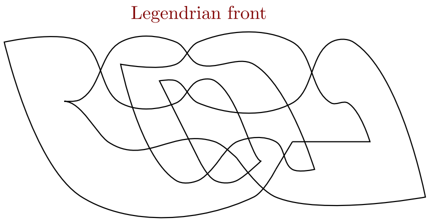

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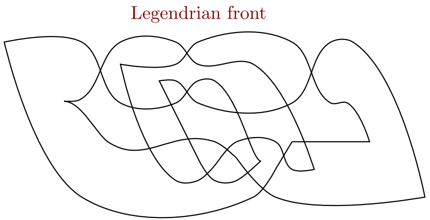

(i) What is the geometric intuition for the moduli $\mathfrak{M}(\Lambda)$? ($\to$ Lagrangian fillings)

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(iii) Why is it useful to have cluster A-structures? ($\to$ Solves several open problems.)

The Main Result gives a fruitful bridge

Use results from cluster algebras to prove results in symplectic topology:

Corollary (with H. Gao, Annals'22)

"Infinitely many Lagrangian fillings."

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Let $u, v \in S_n$ be two permutations, $u \leq v$, and $R(u, v)$ their Richardson variety. Then $\mathbb{C}[R(u, v)]$ is a cluster algebra.

This latter result is a combination of work with D. Weng and separately with E. Gorsky et al. The proof is relatively simple: construct a $\Lambda = \Lambda_{u,v}$ such that $\mathfrak{M}(\Lambda_{u,v}) \cong R(u, v)$ and apply main result.

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Both corollaries are actually stronger, including other Lie types. This has opened fertile ground for more ($\rightarrow$ e.g. AIM Workshop on Jan'23.)

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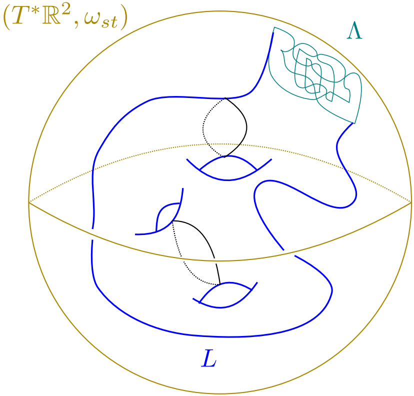

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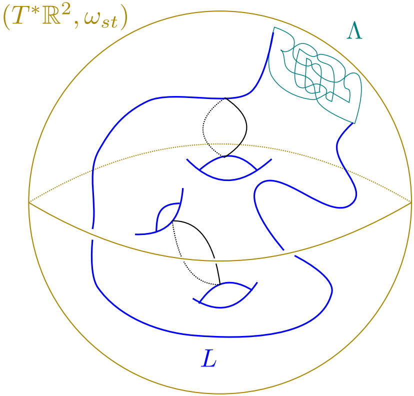

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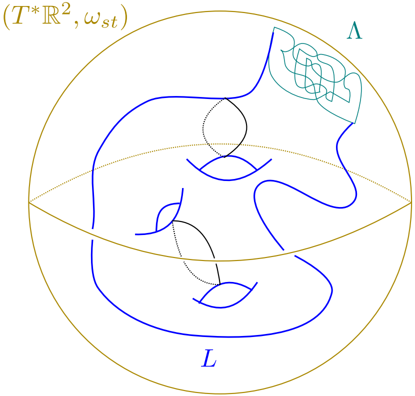

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- Lagrangian filling gives $(\mathbb{C}^*)^{b_1(L)} \subset \mathfrak{M}(\Lambda)$ chart.
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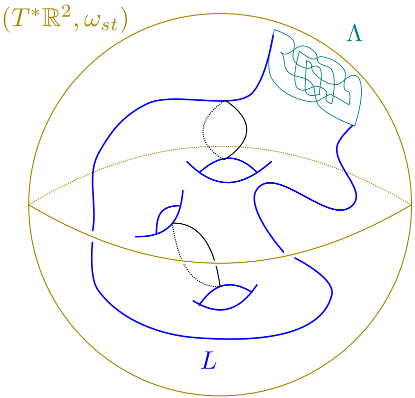
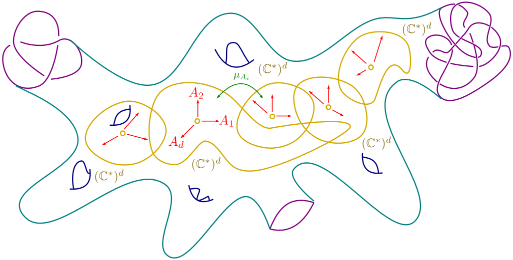


The intuition for cluster varieties

Definition

A **cluster** A**-variety** $\mathfrak{M}$ is a union $\mathfrak{M} \overset{(cd.2)}{=} \bigcup_{s \in S} T_s$, $T_s \cong (\mathbb{C}^*)^d$ algebraic tori, with a **given identification** Spec $T_s \cong \mathbb{C}[A_{s,1}^{\pm 1}, \ldots, A_{s,d}^{\pm 1}]$ such that, in these identifications, **the transition functions are** A**-mutations** $\mu_{A_{s,i}}$.



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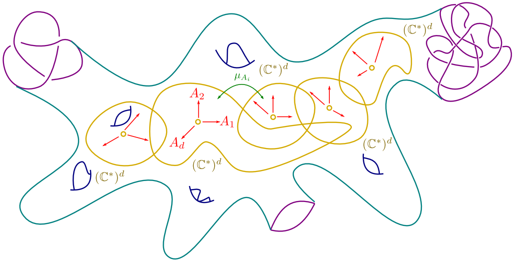

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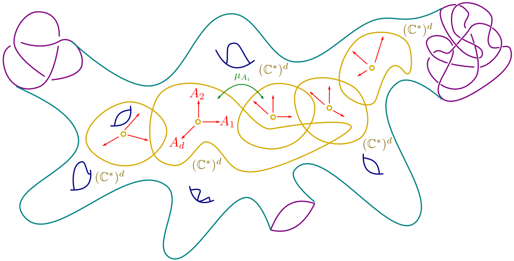

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For us a **Lagrangian filling** gives **toric chart**, but *what does symplectically gives the coordinates $A_{s,j}$ and these transition functions?*

Properties and Examples

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- **Trefoil Example**: Then $\mathfrak{M}(\Lambda_{3_1}) = \{z_1 + z_3 + z_1 z_2 z_3 + 1 = 0\} \subset \mathbb{C}^3$, quiver is $\bullet \to \bullet$ and we have *five* algebraic tori:

$$T_1 = \mathrm{Spec}\{z_1^{\pm 1}, (1+z_1 z_2)^{\pm 1}\}, \quad T_2 = \mathrm{Spec}\{z_3^{\pm 1}, (1+z_3 z_2)^{\pm 1}\}, \quad T_3 = \mathrm{Spec}\{z_1^{\pm 1}, z_3^{\pm 1}\},$$

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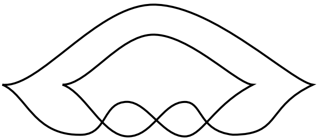
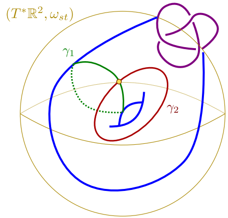
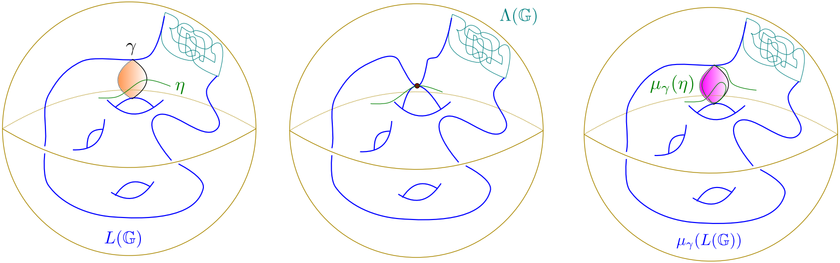




Lagrangian Disk Surgeries

The first symplectic fact towards cluster algebras: **Lagrangian surgery**.



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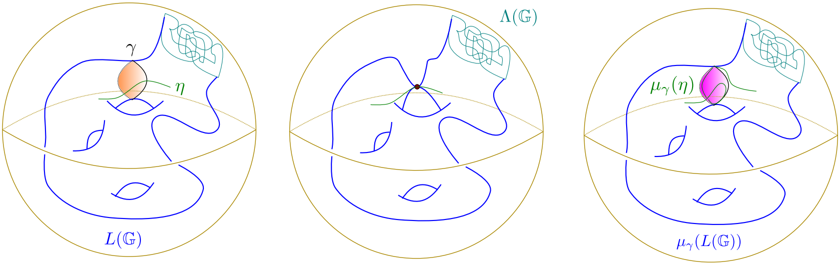

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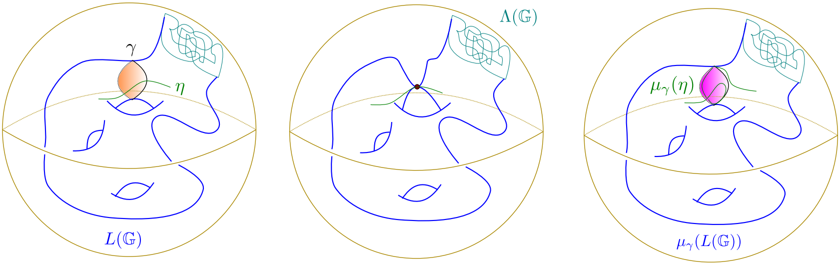

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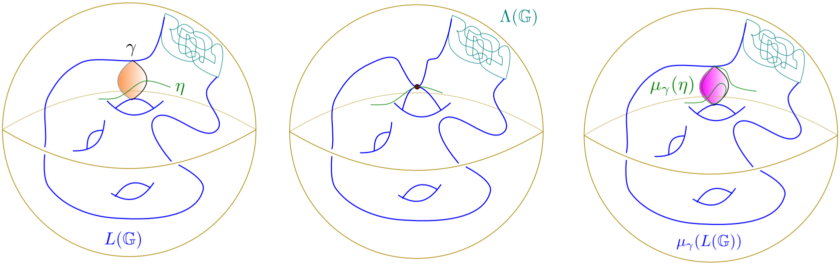

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(iii) How do you find these? $\longrightarrow$ **Legendrian weaves** (*G&T* '22, 116p).
See also "Microlocal Theory of Legendrian Links and Cluster Algebras" (119p).

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(i) $\mathfrak{M}(\Lambda)$ acts as "*space of Lagrangian fillings*", in that an embedded exact Lagrangian $L \subset (\mathbb{R}^4, \lambda_{st})$, $\partial L = \Lambda$, with local system, gives a point in $\mathfrak{M}(\Lambda)$. Focus on Abelian local systems $H^1(L, \mathbb{C}^*)$, then:

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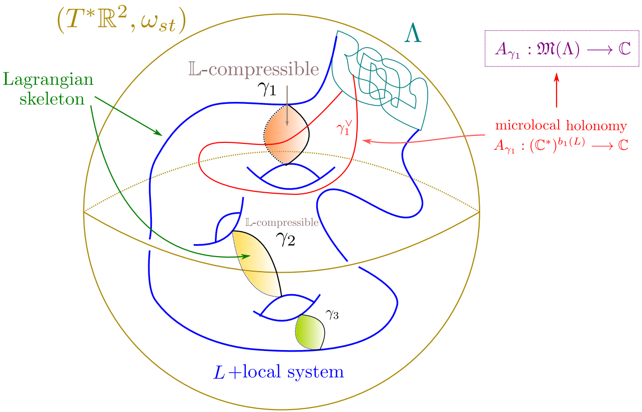
(iii) Need Λ such that D^--stack $\mathfrak{M}(\Lambda)$ is accessible, e.g. affine variety or algebraic quotient thereof, so cluster structures make sense:

$\rightsquigarrow$ Legendrian links Λ from *grid plabic graph* $\mathbb{G}$ or *(-1)-closures of braids*

Summarizing picture

The picture to take home



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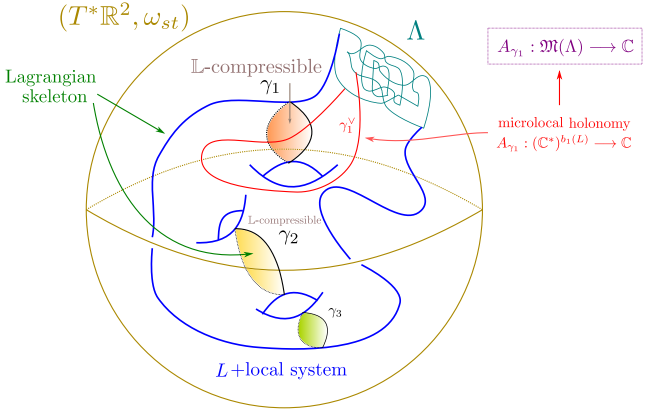

Build $\mathbb{L}$-incompressible system $\rightarrow$ relative Lagrangian skeleton of $(\mathbb{C}^2, \Lambda)$.

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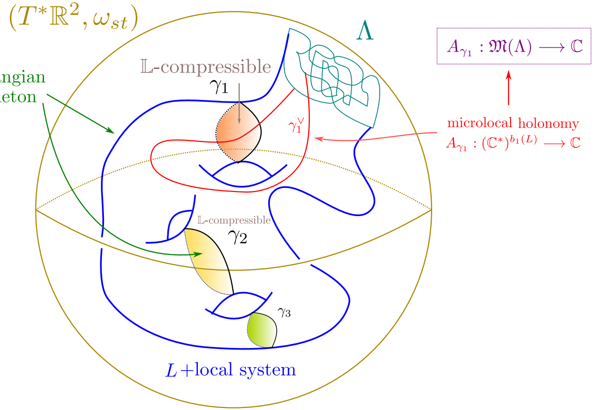

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The special coordinates A_γ are **microlocal holonomies** along dual relative cycles. **Miracle**: they are regular on $\mathfrak{M}(\Lambda)$!

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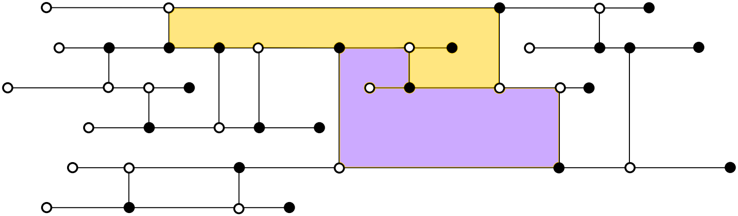
By definition, a **grid plabic graph** $\mathbb{G} \subset \mathbb{R}^2$ is:



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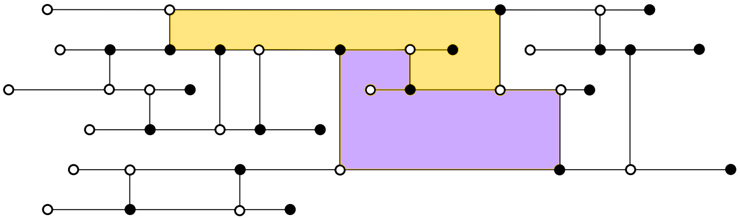
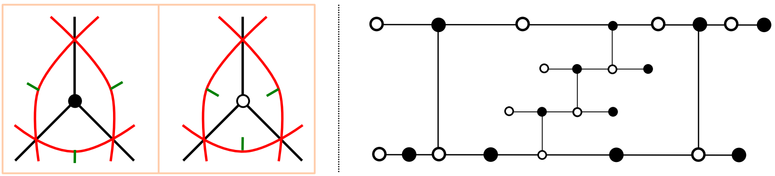


The **alternating strand diagram** associated to $\mathbb{G}$ is drawn as follows:



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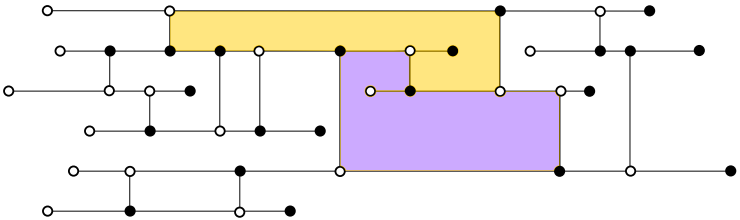

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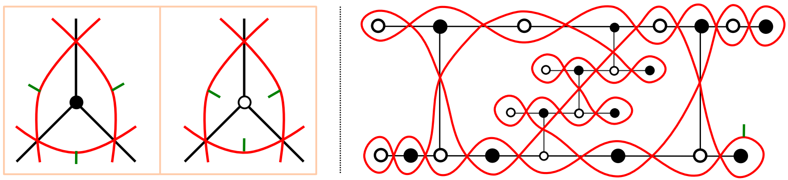

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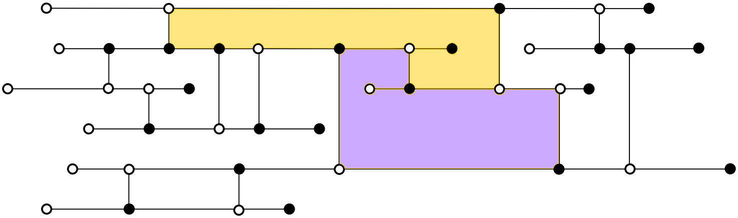

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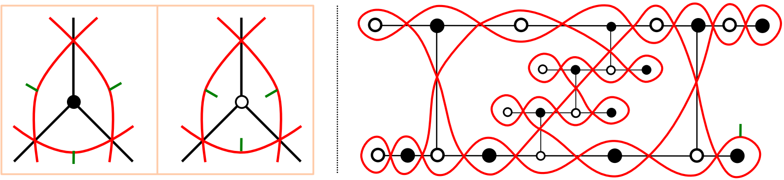

Then, $\Lambda(\mathbb{G}) \subset (\mathbb{R}^3, \xi_{st})$ **is the Legendrian link associated this front**, after satelliting the Legendrian S^1-fiber of $T^*_\infty \mathbb{R}^2$ to the standard unknot.

Examples of $\mathfrak{M}(\Lambda(\mathbb{G}))$

Positive braids: $\mathbb{G}$ plabic fence for $\beta = \sigma_{i_1} \dots \sigma_{i_s} \in \mathrm{Br}_n^+$. Then $\mathfrak{M}(\Lambda(\mathbb{G}))$ is the moduli of tuples of affine flags in $(GL_n/U)^{s+n(n-1)}$ with F_j, F_{j+1} in s_{i_j}-relative position, with a Δ_n^2, plus framing conditions. ([CGGS 1&2])

E.g., for $[\beta] = T(k,n)$, $\mathfrak{M}(\Lambda(\mathbb{G})) \cong \mathrm{Gr}(k, n+k) \setminus \{\Delta_{1,2} \cdots \Delta_{n+k,1} = 0\}$.

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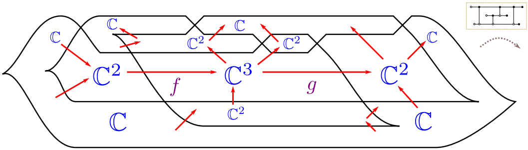

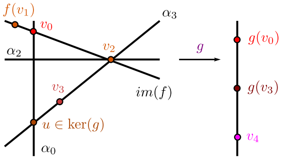

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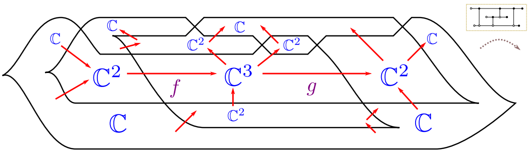

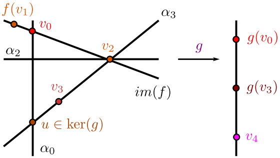

Some degenerations allowed, but **some not**!

The key points for these Legendrians

- **Theorem A**: Let $\mathbb{G}$ be a GP-graph. Then

$$\exists \mathfrak{w}(\mathbb{G}) \text{ weave} \overset{s.t.}{\rightsquigarrow} \text{embedded Lagrangian filling } L(\mathbb{G}) + \text{basis of Y-cycles}$$

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- **Next: Theorem C**. Need to introduce the basis of regular functions:

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  In addition, this **basis** $\mathbb{C}[T_{\mathfrak{w}(\mathbb{G})}]$ must change according to cluster A-mutation for $Q(B(\mathbb{G}))$ when **Lagrangian surgery is performed**.

The microlocal local system on $L(\mathbb{G})$ and $\Lambda(\mathbb{G})$

Define candidate *A*-**variables** with Guillermou-Kashiwara-Schapira maps:

$$\mathbb{I}\mathrm{Sh}_\Lambda(\mathbb{R}^2) \longrightarrow \mu\mathrm{Sh}_\Lambda, \qquad \mu\mathrm{Sh}_\Lambda(\Lambda) \cong \mathrm{Loc}(\Lambda),$$

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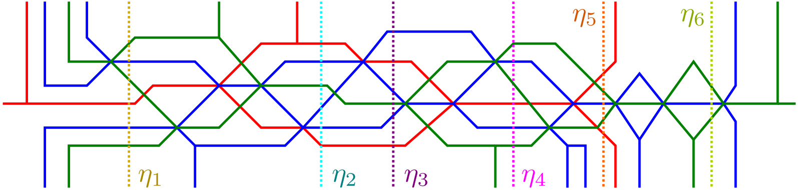
(2) **Theorem**: This parallel transport can be computed by using cones in the braid slice of a weave: *ratios of wedges of decorations*.



Microlocal Merodromies

Definition (Key new concept)

Let $\mathbb{G}$ be a GP-graph and $B(\mathbb{G})$ the **dual relative basis** of Y-cycles of the weave $\mathfrak{w}(\mathbb{G})$. The **microlocal merodromy** along $\eta \in B(\mathbb{G})$ is

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These properties are **not** true unless η belongs to $B(\mathbb{G})$!

The resulting cluster A-structure

Finally, after developing these results, we can conclude:

Theorem (Simplified Upshot)

The moduli $\mathfrak{M}(\mathbb{G})$ admits a **cluster A-structure** *in its coordinate ring, with initial cluster seed* **as symplectically described**.

The crucial step is showing that the inclusion of the upper bound into $\mathfrak{M}(\mathbb{G})$ is an isomorphism, up to codimension 2. This is done by applying "*Technical Properties*" and an argument with *immersed* weaves.

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- The stronger theorem being proved is in great part symplectic geometric: ability to define cluster A-coordinate symplectically via **merodromies** on

Lagrangian fillings and a **basis of dually $\mathbb{L}$-compressible relative cycles.**

The end

Thanks a lot!

"But this is the simplified version for the general public."