Moduli spaces of nodal curves from homotopical algebra

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Outline

1. Background
   - Motivation from mirror symmetry
   - Properads
2. Results
   - Deligne-Mumford compactifications
   - Partial compactifications
3. Further Directions

Motivation from mirror symmetry

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Categorical enumerative invariants: For suitable $\mathcal{C} \rightsquigarrow CEI(\mathcal{C})$.

Motivation from mirror symmetry

- Costello: $\mathcal{C}$ compact CY, A_∞ category,

$$H_\bullet(\mathcal{M}^{fr}_{g,k,l}) \otimes HH_\bullet(\mathcal{C})^{\otimes k} \to HH_\bullet(\mathcal{C})^{\otimes l}, k \geq 1. \qquad (\dagger)$$

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- Kontsevich: Under suitable conditions these maps extend to

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Question

When does ($\dagger$) *induce maps* ($\dagger\dagger$)*?*

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Properads

Definition

A properad P (in topological spaces) consists of

- space of operations $P(k, l)$ for every $k, l \geq 0$
- composition maps

$$P(k, l) \times P(m, n) \to P(k + m - s, n + l - s),$$

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- $\mathcal{M}^{fr}$

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In addition, we also include exceptional curves as follows

$$\overline{\mathcal{M}}(1,1) := *, \quad \overline{\mathcal{M}}(0,2) := *, \quad \overline{\mathcal{M}}(2,0) := *,$$

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Properads

Examples:

- $\mathcal{M}^{fr}$
- $\overline{\mathcal{M}}$
- For any space X, Endomorphism properad $End(X)$ with

$$End(X)(k,l) = Maps(X^k, X^l).$$

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($\dagger$) gives an action of $H_\bullet(\mathcal{M}^{fr})$ on $HH_\bullet(\mathcal{C})$

Question (Restated)

When does the action of $H_\bullet(\mathcal{M}^{fr})$ on $HH_\bullet(\mathcal{C})$ induce an action of $H_\bullet(\overline{\mathcal{M}})$?

Outline

Deligne-Mumford compactifications

Theorem (D.)

$\overline{\mathcal{M}}$ is the homotopy pushout of the diagram

$$
\begin{tikzcd}
\{\mathcal{M}^{fr}_{0,1,1} \quad \mathcal{M}^{fr}_{0,0,2} \quad \mathcal{M}^{fr}_{0,2,0} \quad \mathcal{M}^{fr}_{0,0,1} \quad \mathcal{M}^{fr}_{0,0,1}\} \arrow[r] \arrow[d] & \{* \quad * \quad * \quad * \quad *\} \\
\mathcal{M}^{fr} &
\end{tikzcd}
$$

in the category of io-properads.*

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Deligne-Mumford compactifications

At the level of algebras

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\mathcal{M}^{fr} \arrow[r, bend right] & End(A)
\end{tikzcd}$$

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 & & \mathcal{M}^{fr} \arrow[rrrrr] \arrow[drrrrr, bend right] & & & & & \overline{\mathcal{M}} \arrow[d, red] & & \\
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N

true on nose! Only up to homotopy coherences.

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- Homotopy coherent statement

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- At level of moduli spaces/stacks, not just chains or homology

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$\overline{\mathcal{M}}$ *is the homotopy pushout of the diagram*

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- Curves of all genera, moduli spaces with multiple inputs, outputs

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 & & \big\downarrow & & & & \\
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- * *io*-properad a modifications of properad omitting operations in arity $(0, 0)$

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- For X an exact symplectic manifold with cylindrical ends, symplectic cohomology $SH^{\bullet}(X)$ is expected to carry a (chain-level) action of properad of Riemann surfaces.

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- Action of S^1-family from $\mathcal{M}^{fr}_{0,0,2}$ is always trivialized.

Partial Compactifications

$\widehat{\mathcal{M}}_{g,k,l}$: moduli space of stable nodal surfaces with cylindrical input and output ends, such that each irreducible component contains an output.

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& \mathcal{M}^{fr}_{l\geq 1} & & & &
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Secondary operations and Rabinowitz Floer cohomology

Relative space $(\widehat{\mathcal{M}}_{g,k,l}, \partial\widehat{\mathcal{M}}_{g,k,l})$ natural space for classes parametrizing secondary operations:

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Conjecture

Algebraic structure of Rabinowitz Floer Cohomology:

- *$SC^\bullet(X)$ carries a (chain-level) $C_\bullet(\widehat{\mathcal{M}})$-action and $SC_\bullet(X)$ is a module over $\widehat{\mathcal{M}}$-algebra $SC^\bullet(X)$.*
- *Continuation map $c\colon SC_\bullet(X) \to SC^\bullet(X)$ is a map of modules and hence the $\widehat{\mathcal{M}}$-action descends to the cone*

  $$RFC^\bullet(X) = Cone(c\colon SC_\bullet(X) \to SC^\bullet(X))$$

  computing Rabinowitz cohomology.
- *The point class in $C_\bullet(\widehat{\mathcal{M}}_{0,0,2})$, and hence the ideal $\partial\widehat{\mathcal{M}}$ generated by it, act trivially on $RFC^\bullet(X)$. Thus there is induced action of relative chains on $(\widehat{M}, \partial\widehat{M})$.*