Non-Weinstein Liouville domains and three-dimensional Anosov flows

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Based on

- Kai Cieliebak, Oleg Lazarev, T. M., and Agustin Moreno. *Floer theory of Anosov flows in dimension three.* arXiv:2211.07453
- T. M. *Anosov flows and Liouville pairs in dimension three.* arXiv:2211.11036

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*A **Liouville domain** is (V, ω, λ), where*

- *V compact with boundary $\partial V = M$,*
- *ω symplectic,*
- *$\omega = d\lambda$, $\omega(Z, \cdot) = \lambda$, Z pos. transverse to ∂M ($\lambda_{|M}$ contact).*

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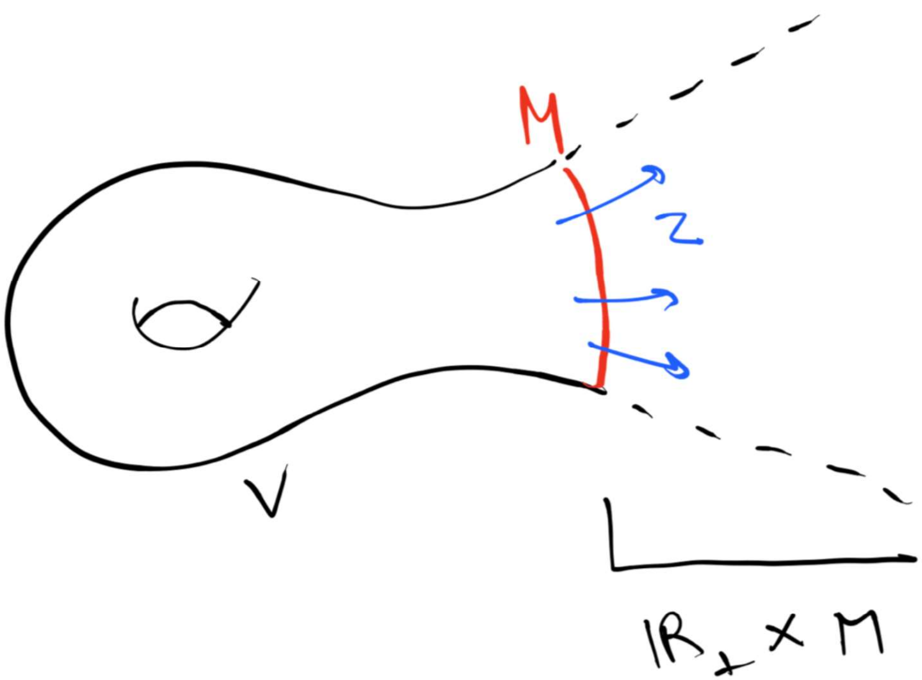

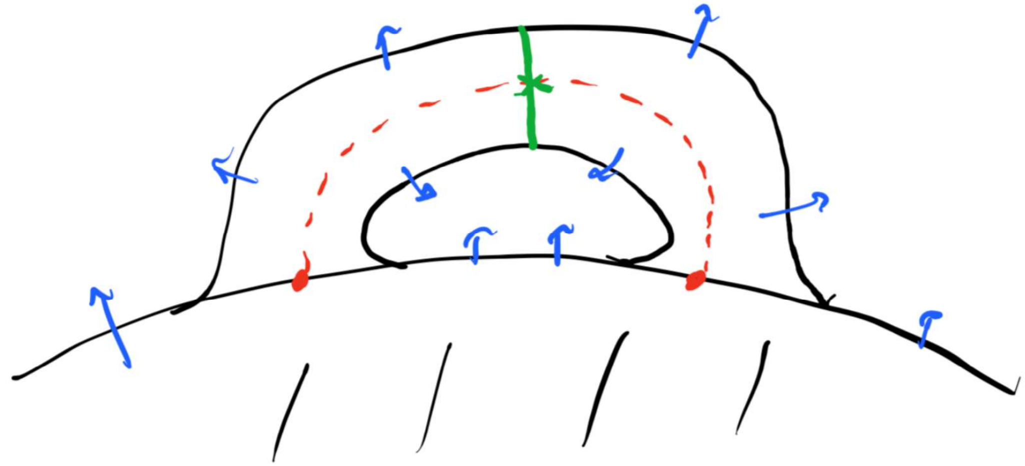

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$$\omega := \omega_{\mathsf{can}} + \pi^*\sigma$$

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Magic: on $\widehat{V} = T^*\Sigma \backslash 0_\Sigma$,

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$M_\pm = \{\pm 1\} \times M$, $\alpha_\pm = \lambda_{|M_\pm}$,

$$\alpha_- = \alpha_{\text{pre}}, \qquad \alpha_+ = \alpha_{\text{can}}.$$

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On $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$,

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$A \in \mathrm{SL}(2,\mathbb{Z})$, $\mathrm{tr}(A) > 2$. Write $D = PAP^{-1}$, $P \in \mathrm{SL}(2,\mathbb{R})$,

$$D = \begin{pmatrix} e^\nu & 0 \\ 0 & e^{-\nu} \end{pmatrix}$$

$\rightsquigarrow \alpha_\pm$ induce 1-forms on $M =$ suspension of $A : \mathbb{T}^2 \circlearrowleft$, get Liouville structure on $\mathbb{R} \times M$.

Non-Weinstein Liouville domains : Lagrangian submanifolds

Question

In the McDuff and torus bundle domains, are there interesting

- *Closed exact Lagrangians ($\lambda_{|L} = df$)?*
- *Closed weakly exact Lagrangians ($\omega \cdot \pi_2(M, L) = 0$)?*
- *Non-compact exact Lagrangians, cylindrical at infinity (Z tangent to L outside compact)?*

Non-Weinstein Liouville domains : Lagrangian submanifolds

Theorem (CLMM 2022)

In McDuff domain/manifold,

- $\gamma \subset \Sigma$ *closed geodesic* $\rightsquigarrow$ *exact Lagrangian torus* $\mathbb{T}_\gamma$.
- *Similarly, get non-exact, weakly exact tori.*
- γ *oriented* $\rightsquigarrow$ *positive conormal lift* $L_\gamma \subset T^*\Sigma \backslash 0_\Sigma$ *exact cylindrical Lagrangian.*

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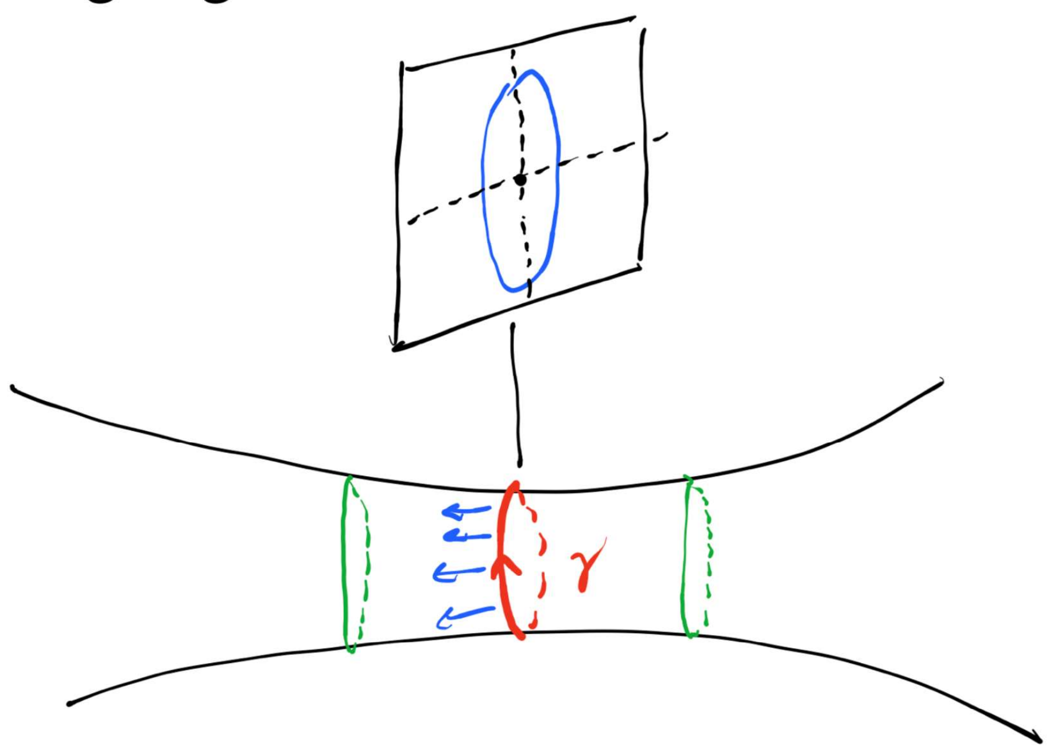

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Theorem (CLMM 2022)

In torus bundle domain/manifold,

- *No closed exact (orientable) Lagrangians.*
- $\mathbb{T}^2$*-fibers of* $\mathbb{R} \times M \to \mathbb{R} \times S^1$ *are weakly exact Lagrangians.*
- $\mathcal{O} \subset \mathbb{T}^2$ *periodic orbit of* $A \rightsquigarrow$ *exact cylindrical Lagrangian* $L_{\mathcal{O}} \subset \mathbb{R} \times M$ *of the form* $\mathbb{R} \times \Lambda_{\mathcal{O}}$, $\Lambda_{\mathcal{O}} \subset M$ *suspension of* $\mathcal{O}$.

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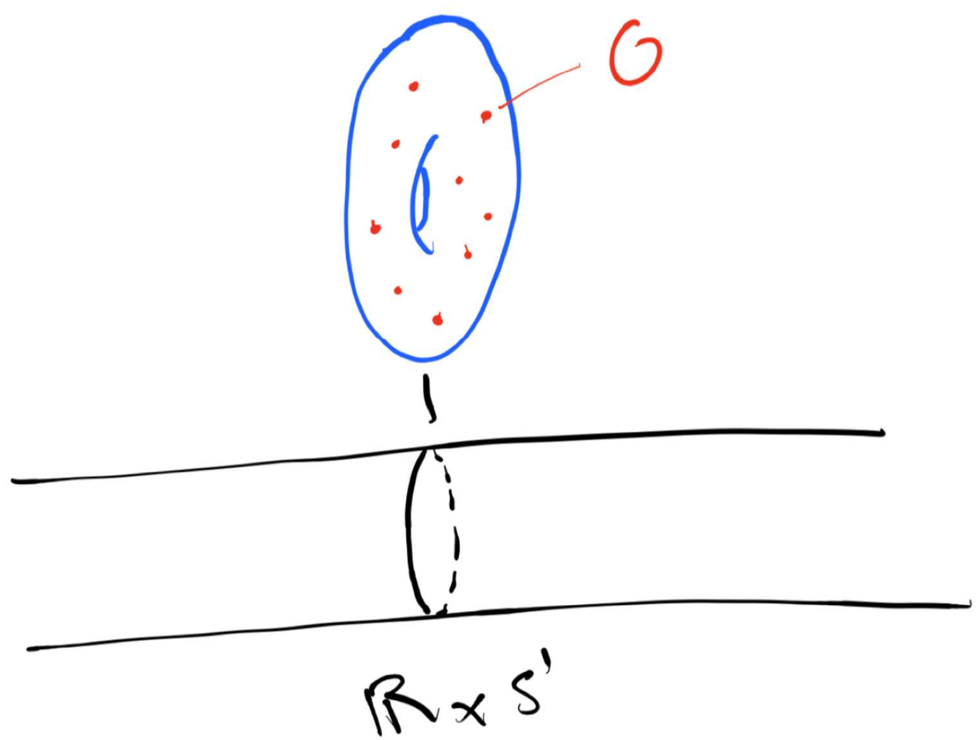

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Both McDuff and torus bundle manifolds are of the form $\mathbb{R} \times M$,

$$\lambda = e^{-s}\alpha_- + e^s\alpha_+,$$

Z satisfies $\alpha_\pm(Z) = 0$, skeleton is $M_0 = \{0\} \times M$, Z tangent to M_0 and restricts to an **Anosov flow**.

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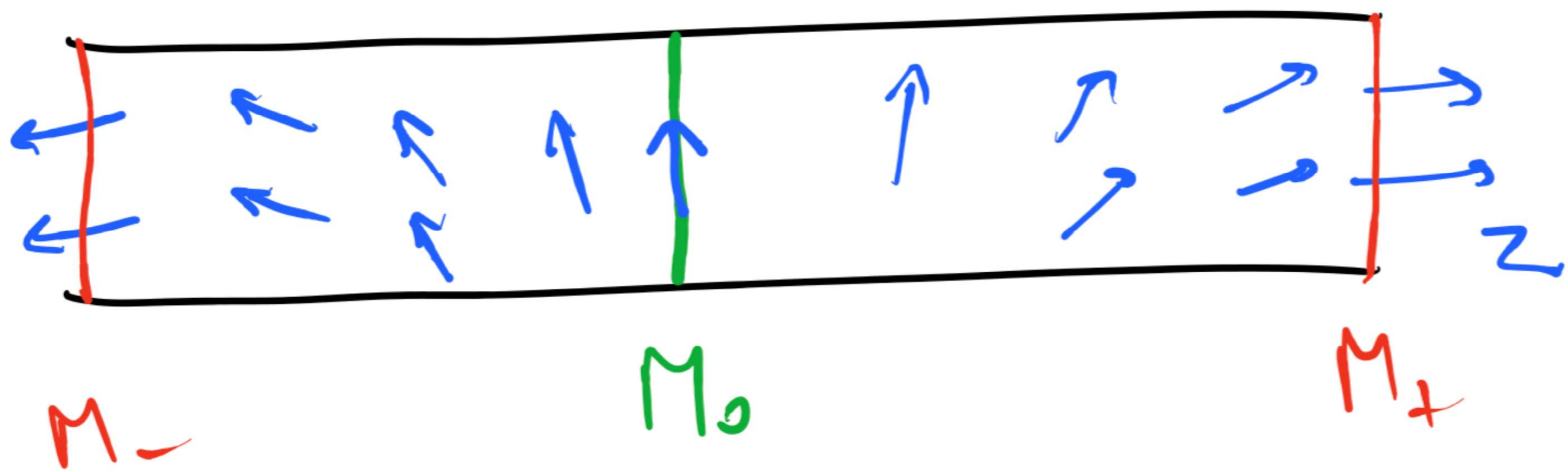

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*M^3 closed oriented, $\{\phi^t\}$ non-singular flow generated by vector field X is **Anosov** if $\exists$ $\mathcal{C}^0$ invariant splitting*

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Integrate to *taut foliations* $\mathcal{F}^{ws/wu}$.

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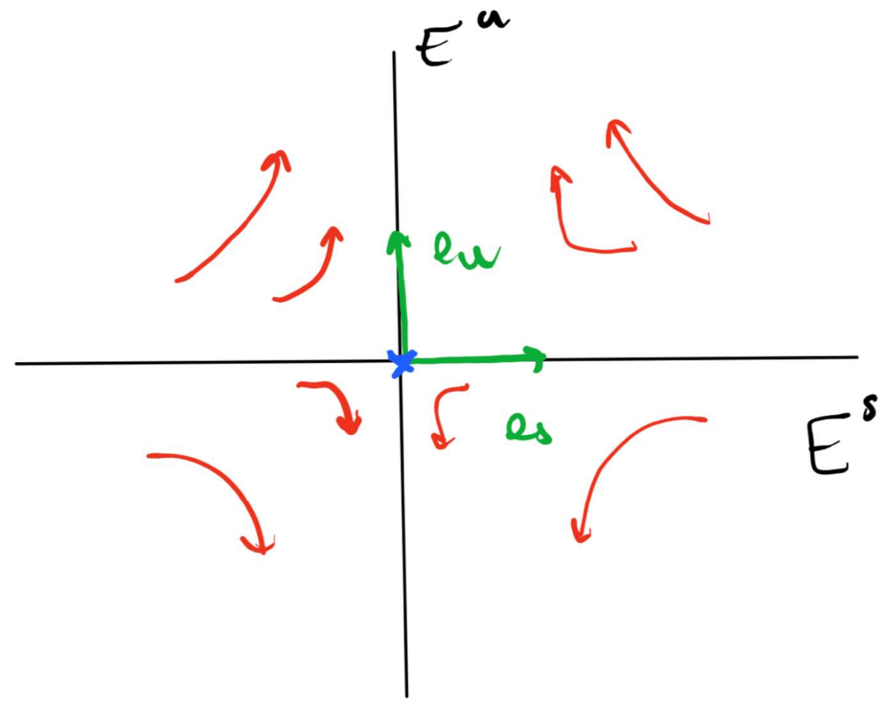

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1. *$\alpha_\pm$ contact forms,*
2. *$\xi_\pm = \ker \alpha_\pm$ are transverse,*
3. *$\xi_- \cap \xi_+ = \langle X \rangle$, X Anosov vector field on M.*

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Notations:

- $\mathcal{AL}$: space of AL structures on $\mathbb{R} \times M$,
- $\mathcal{AF}$: space of Anosov flows on M up to positive time reparametrization.
- $\mathcal{I} : \mathcal{AL} \to \mathcal{AF}, (\alpha_-, \alpha_+) \mapsto$ "$\xi_- \cap \xi_+$".

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Corollary

Anosov flow on $M \rightsquigarrow$ Liouville structure on $\mathbb{R} \times M$, well-defined up to homotopy, only depends on homotopy class of Anosov flow. Symplectic invariants (SH^*, $\mathcal{W}Fuk$, etc.) are ***invariants of the flow****.*

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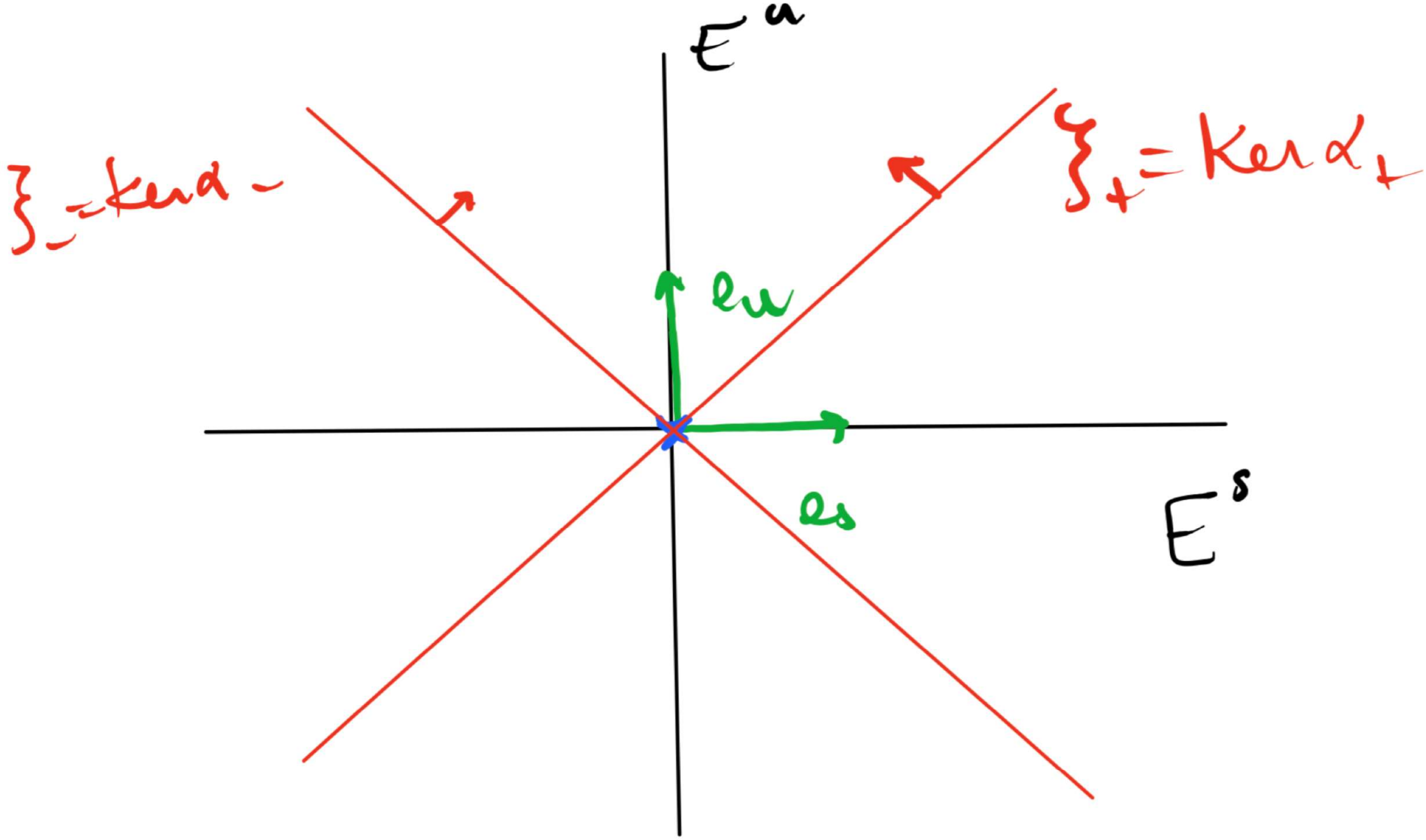

$$\ker \alpha_{s/u} = E^{wu/ws}, \qquad \mathcal{L}_X \alpha_{s/u} = r_{s/u} \alpha_{s/u}, \qquad r_s < 0 < r_u.$$

$$\alpha_{\pm} := \alpha_u \mp \alpha_s$$

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$$L_\Lambda = \mathbb{R} \times \Lambda \subset \mathbb{R} \times M$$

is **strictly exact** : $\lambda_{|L_\Lambda} \equiv 0$. Recall : $\lambda = e^{-s}\alpha_- + e^{s}\alpha_+$.

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Question

Algebraic structure of $\mathcal{W}_0$? Does $\mathcal{W}_0$ split-generate $\mathcal{W}Fuk$?

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Nevertheless: still possible that $\mathcal{W}_0$ split-generates $\mathcal{W}Fuk$...

Thank you for your attention!