Non-Weinstein Liouville domains and three-dimensional Anosov flows

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Symplectic Zoominar, 11/25/2022

Based on

Liouville and Weinstein domains
Liouville and Weinstein domains

Definition
A Liouville domain is \((V, \omega, \lambda)\), where

- \(V\) compact with boundary \(\partial V = M\),
- \(\omega\) symplectic,
- \(\omega = d\lambda, \omega(Z, \cdot) = \lambda, Z\) pos. transverse to \(\partial M\) (\(\lambda|_M\) contact).
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**Weinstein domain**: \(\exists \phi : V \rightarrow \mathbb{R}\) Morse-Lyapunov function for \(Z\).
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**McDuff (1991):** $\Sigma$ closed hyperbolic surface, $\sigma$ volume form, symplectic form on $T^*\Sigma$:

$$\omega := \omega_{\text{can}} + \pi^* \sigma$$

where

$$\omega_{\text{can}} = \sum_i dp_i \wedge dq_i, \quad \pi : T^*\Sigma \to \Sigma.$$
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Magic: on $\hat{V} = T^*\Sigma \setminus 0_\Sigma$,

$$\omega = d\lambda,$$

$\leadsto (V, \omega, \lambda)$ Liouville domain! $V \cong [-1, 1] \times S^*\Sigma$ not Weinstein.
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$M_\pm = \{\pm 1\} \times M$, $\alpha_\pm = \lambda|_{M_\pm}$,

$$\alpha_- = \alpha_{\text{pre}}, \quad \alpha_+ = \alpha_{\text{can}}.$$
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Geiges (1995): on $\mathbb{R}^3$, 

$$\alpha_\pm := \pm e^z \, dx + e^{-z} \, dy.$$ 

On $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, 

$$\lambda = e^{-s} \alpha_- + e^s \alpha_+.$$ 

\[\Rightarrow\] $\omega = d\lambda$ symplectic.
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$A \in \text{SL}(2, \mathbb{Z}), \text{tr}(A) > 2$. Write $D = PAP^{-1}, P \in \text{SL}(2, \mathbb{R})$,

$$D = \begin{pmatrix} e^\nu & 0 \\ 0 & e^{-\nu} \end{pmatrix}.$$ 

$\implies \alpha_\pm$ induce 1-forms on $M = \text{suspension of } A : \mathbb{T}^2 \hookrightarrow$, get Liouville structure on $\mathbb{R} \times M$. 
Non-Weinstein Liouville domains: Lagrangian submanifolds

Question

In the McDuff and torus bundle domains, are there interesting

- Closed exact Lagrangians ($\lambda|_L = df$)?
- Closed weakly exact Lagrangians ($\omega \cdot \pi_2(M, L) = 0$)?
- Non-compact exact Lagrangians, cylindrical at infinity (Z tangent to $L$ outside compact)?
Non-Weinstein Liouville domains: Lagrangian submanifolds

Theorem (CLMM 2022)

In McDuff domain/manifold,

- $\gamma \subset \Sigma$ closed geodesic $\rightsquigarrow$ exact Lagrangian torus $\mathbb{T}_\gamma$.
- Similarly, get non-exact, weakly exact tori.
- $\gamma$ oriented $\rightsquigarrow$ positive conormal lift $L_\gamma \subset T^*\Sigma \setminus 0\Sigma$ exact cylindrical Lagrangian.
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Theorem (CLMM 2022)

In torus bundle domain/manifold,

- No closed exact (orientable) Lagrangians.
- $\mathbb{T}^2$-fibers of $\mathbb{R} \times M \to \mathbb{R} \times S^1$ are weakly exact Lagrangians.
- $\mathcal{O} \subset \mathbb{T}^2$ periodic orbit of $A \hookrightarrow$ exact cylindrical Lagrangian $L_\mathcal{O} \subset \mathbb{R} \times M$ of the form $\mathbb{R} \times \Lambda_\mathcal{O}$, $\Lambda_\mathcal{O} \subset M$ suspension of $\mathcal{O}$. 
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\[ \begin{align*} \mathbb{R} \times S' \end{align*} \]
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Both McDuff and torus bundle manifolds are of the form $\mathbb{R} \times M$,

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$Z$ satisfies $\alpha_\pm(Z) = 0$, skeleton is $M_0 = \{0\} \times M$, $Z$ tangent to $M_0$ and restricts to an **Anosov flow**.
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$M^3$ closed oriented, $\{\phi^t\}$ non-singular flow generated by vector field $X$ is Anosov if $\exists C^0$ invariant splitting

$$TM = \langle X \rangle \oplus E^s \oplus E^u,$$
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$E^{ws/wu} = \langle X \rangle \oplus E^{s/u}.$
Integrate to taut foliations $\mathcal{F}^{ws/wu}.$
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1. $\alpha_\pm$ contact forms,
2. $\xi_\pm = \ker \alpha_\pm$ are transverse,
3. $\xi_- \cap \xi_+ = \langle X \rangle$, $X$ Anosov vector field on $M$. 
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Notations:

- $\mathcal{AL}$: space of AL structures on $\mathbb{R} \times M$,
- $\mathcal{AF}$: space of Anosov flows on $M$ up to positive time reparametrization.
- $I: \mathcal{AL} \rightarrow \mathcal{AF}$, $(\alpha_-, \alpha_+) \mapsto \langle \xi_-, \xi_+ \rangle$. 
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Theorem (M. 2022)

$\mathcal{I}$ is an acyclic Serre fibration, hence a homotopy equivalence.
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Corollary
Anosov flow on $M$ $\mapsfrom$ Liouville structure on $\mathbb{R} \times M$, well-defined up to homotopy, only depends on homotopy class of Anosov flow. Symplectic invariants ($SH^*$, $WFuk$, etc.) are invariants of the flow.
Anosov Liouville manifolds: construction
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\[ \ker \alpha_{s/u} = E^{wu/ws}, \quad \mathcal{L}_X \alpha_{s/u} = r_{s/u} \alpha_{s/u}, \quad r_s < 0 < r_u. \]

\[ \alpha_\pm := \alpha_u \mp \alpha_s \]
Anosov Liouville manifolds: Lagrangian cylinders
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Since $\alpha_{\pm}(X) = 0$, closed orbit $\Lambda$ of $X$ is Legendrian for $\xi_{\pm}$, and

$$L_{\Lambda} = \mathbb{R} \times \Lambda \subset \mathbb{R} \times M$$

is strictly exact: $\lambda|_{L_{\Lambda}} \equiv 0$. Recall: $\lambda = e^{-s} \alpha_- + e^{s} \alpha_+$. 
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- $L_\gamma$ positive conormal lift of oriented closed geodesic $\gamma \subset \Sigma$ in McDuff domain,
- $L_\Omega$ cylinder over suspension of closed orbit of $A$ in torus bundle domain.
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$\{L_\Lambda\}$ spans $\mathcal{W}_0 \subset \mathcal{W}Fuk$, the **orbit category**.
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**Question**

*Algebraic structure of $\mathcal{W}_0$? Does $\mathcal{W}_0$ split-generate $\mathcal{W}\text{Fuk}$?*
Anosov Liouville manifolds: (non)-split-generation
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Theorem (CLMM 2022)

$\mathcal{W}_0 \subset \mathcal{W}_{Fuk}$ does not satisfy Abouzaid’s criterion:

$$\mathcal{O}_C_0 : HH_*(\mathcal{W}_0) \to SH^{*+2}(V)$$

does not hit the unit. It has nontrivial coker are infinite rank ker.
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\( \mathcal{W}_0 \subset \mathcal{W}_{Fuk} \) is “maximally non-finitely split-generated”: if \( \mathcal{A} \subset \mathcal{W}_0 \) and \( L \in \mathcal{W}_0 \setminus \mathcal{A} \), then \( L \) not split-generated by \( \mathcal{A} \).
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\[ \mathcal{W}_0 \subset \mathcal{WFuk} \text{ is “maximally non-finitely split-generated”: if } A \subset \mathcal{W}_0 \text{ and } L \in \mathcal{W}_0 \setminus A, \text{ then } L \text{ not split-generated by } A. \text{ This implies:} \]
- \[ L^\wedge \not\cong L^{\wedge'}, \text{ for } \wedge \neq \wedge', \]
- \[ \mathcal{W}_0 \text{ not finitely split-generated,} \]
- \[ \mathcal{W}_0 \text{ not homologically smooth.} \]
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**Theorem (CLMM 2022)**

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**Theorem (CLMM 2022)**

$\mathcal{W}_0 \subset \mathcal{W}Fuk$ is “maximally non-finitely split-generated”:

if $A \subset \mathcal{W}_0$ and $L \in \mathcal{W}_0 \setminus A$, then $L$ not split-generated by $A$. This implies:

- $L_\Lambda \not\cong L_{\Lambda'}$ for $\Lambda \neq \Lambda'$,
- $\mathcal{W}_0$ not finitely split-generated,
- $\mathcal{W}_0$ not homologically smooth.

**Nevertheless:** still possible that $\mathcal{W}_0$ split-generates $\mathcal{W}Fuk$...
Thank you for your attention!