Periodic orbits and Birkhoff sections of Stable Hamiltonian structures

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Throughout the talk:

$M =$ orientable closed three-manifold
$X =$ non-vanishing smooth vector field preserving some volume form μ

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In S^3 it is not known if X always admits a periodic orbit. It does not if X is only C^1 as shown by Kuperberg '96.

Motivation

- (W, ω) a four-dimensional symplectic manifold. Given $H \in C^\infty(M)$, let X_H be the Hamiltonian vector field. If $M = H^{-1}(c)$ where c is regular then $X = X_H|_M$ is non-vashing and volume-preserving in M.

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Remark

A vector field X is Eulerisable if there exists a metric for which X is a stationary solution to the Euler equations. Reeb fields defined by contact forms and by stable Hamiltonian structures are Eulerisable (Sullivan, Etnyre-Ghrist, Rechtman, Cieliebak-Volkov).

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If M is a compact with boundary, a global section is an embedded surface with boundary Σ satisfying $\partial\Sigma \subset \partial M$.

Birkhoff sections

Definition

A Birkhoff section of X *is an immersed compact surface with boundary* Σ *satisfying:*

1. *its interior is embedded and transverse to* X*,*
2. *its boundary is mapped to periodic orbits of* X*,*
3. *there exists some* $T > 0$ *such that for each* $p \in M$*, the flow segment* $\varphi_{[0,T]}(p)$ *intersects* Σ*.*

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Some classes of flows with Birkhoff sections: geodesic flows on positively curved spheres and negatively curved surfaces (Birkhoff '17), transitive Anosov flows (Fried '83), transitive pseudo-Anosov flows (Brunella '95, see also Tsang 2022).

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Techniques of Birkhoff for more general geodesic flows (Contreras –Knieper–Mazzucchelli–Schulz 2022).

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- Two or infinitely many for nondegenerate Reeb flows (Colin-Dehornoy-Rechtman '20)
- Complete understanding of Reeb flows with two periodic orbits (Cristofaro-Gardiner–Hryniewicz-Hutchings-Liu '21, Hutchings-Taubes '09)

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- Adapted broken book decomposition for nondegenerate Reeb flows (Colin-Dehornoy-Rechtman '20)
- Strongly nondegenerate contact forms admit a Birkhoff section (Contreras-Mazzucchelli '21)
- Open and dense set of contact forms admits a Birkhoff section on any three-manifold (Colin-Dehornoy-Hryniewicz-Rechtman '22)

Stable Hamiltonian structures

First defined by Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder (2003), foundations in 3D by Cieliebak-Volkov (2015).

Definition

A stable Hamiltonian structure is a pair $(\lambda, \omega) \in \Omega^1(M) \times \Omega^2(M)$ *such that:*

- $\lambda \wedge \omega > 0$,
- $d\omega = 0$,
- $\ker \omega \subseteq \ker d\lambda$.

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Given a volume-preserving vector field X, the following are equivalent:

1. X is the Reeb field of a SHS
2. X preserves some transverse plane field
3. there is a metric on M making X of unit length and the flowlines geodesics

Introduced by Hofer-Zehnder (1994). Identified with the previous definition by Eliashberg-Kim-Polterovich and Cieliebak-Mohnke around 2005.

Definition

A hypersurface M on a symplectic four-manifold (W, ω) is **stable** *if there exists a tubular neighborhood $U \cong M \times (-\varepsilon, \varepsilon)$ such that the characteristic foliations of $M \times \{t\}$ are all conjugate via a family of diffeomorphisms depending smoothly on t.*

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A steady solution to the Euler equations of Beltrami type is the (reparametrized) Reeb field of a SHS.

Concrete examples

Contact forms. Given α a contact form, the pair $(\alpha, d\alpha)$ defines a stable Hamiltonian structure.

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Suspension flows. Given an area-preserving diffeomorphism of a surface $f : \Sigma \to \Sigma$, it induces a stable Hamiltonian structure (λ, ω) with $d\lambda = 0$ on the suspended manifold $M = \Sigma \times [0, 1]/\sim$. The flow admits a global cross section.

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Reeb flows with a first integral. Let α be a contact form defining a Reeb field X with a first integral $g \in C^\infty(M)$, that we assume positive. Then $(\alpha, gd\alpha)$ defines a SHS whose Reeb field is the Reeb field of α.

General picture

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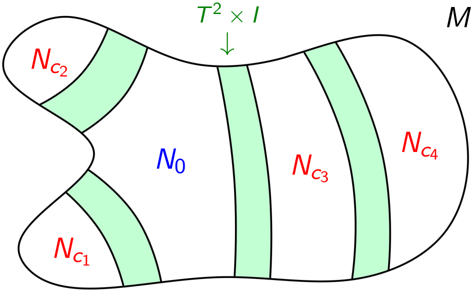
Let $N_{c_i} = f^{-1}[c_i - \delta, c_i + \delta]$ for each singular value c_i of f.



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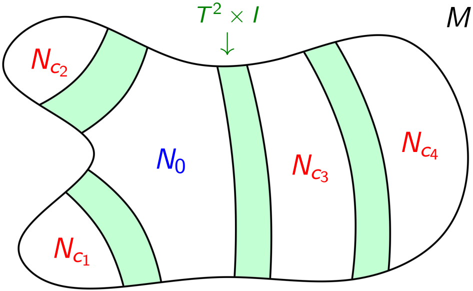

In each integrable region $U_i \cong T^2 \times I$, the flow is fiberwise linear, the "slope" of X is constant (rational or irrational) or non-constant.

The Weinstein conjecture

Theorem (Hutchings-Taubes '09, Rechtman '10 (some cases))

Let M be a closed three-manifold that is not a torus bundle over S^1. Then any Reeb field of any SHS on M admits a closed orbit.

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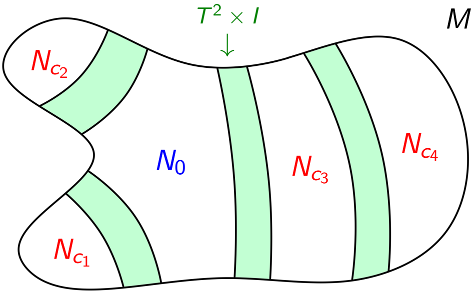
- *M is a three-torus or a positive parabolic bundle and X is orbit equivalent to the suspension of an aperiodic symplectomorphism of the two-torus,*
- *M is a hyperbolic bundle and X does* **not** *admit a global section, but after cutting open along an invariant tori the flow is orbit equivalent to the suspension of a pseudorotation of the closed annulus.*

$N_{cont} = \bigsqcup_{c_i \neq 0} N_{c_i}$ is called the "contact region".



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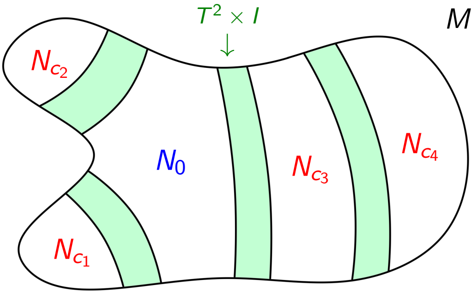

Theorem (Cieliebak-Volkov '15)

The flow in N_0 admits a global section (a surface with boundary).

Hence we call N_0 the "suspension region".

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Definition

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Theorem (Cieliebak-Volkov '15)

Contact non-degenerate SHS are C^1-dense in the set of stable Hamiltonian structures of M.

"Two or infinitely many"

Theorem

Let (λ, ω) be a contact non-degenerate SHS with at least one periodic orbit. It has infinitely many periodic orbits unless:

- *the flow orbit equivalent to the suspension of a symplectomorphism of a surface Σ_g with finitely many periodic points.*

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Remark

It follows from the proof that the degenerate case would follow from a proof for (contact) Reeb fields.

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Corollary

On any closed three-manifold, there exists a C^1-dense, C^2-open set of SHS whose Reeb field admits a Birkhoff section.

Concretely, given any SHS, it is exact stable homotopic to a C^1-close SHS with a Birkhoff section. Cieliebak-Volkov (2014) showed that any SHS is stable homotopic to one supported by an open book decomposition.

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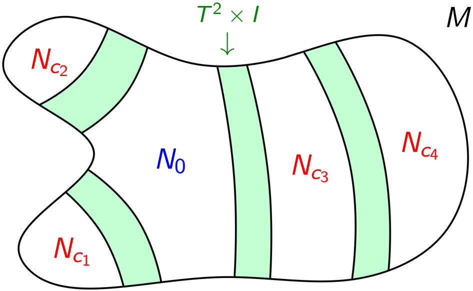
3. Case of interest: f is non-constant.



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Proposition

Let X be a non-degenerate Reeb vector field in a three-manifold with boundary. Assume that near the boundary it is foliated by irrational invariant tori, and that it has finitely many periodic orbits. Then

- $M \cong D^2 \times S^1$ *and X is the suspension of an irrational pseudorotation of the disk,*
- $M \cong T^2 \times I$ *and X is the suspension of an irrational rotation of the annulus.*

Each connected component of the "contact region" is as above.

For the N_0 region:

Theorem

Let $\varphi : \Sigma \to \Sigma$ be a symplectomorphism of a surface with boundary. Assume that it has no periodic points in the boundary. Then it has periodic points of arbitrarily large period unless:

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We have decomposed our manifold as a union of $T^2 \times I$ and $D^2 \times S^1$. There is at least one $D^2 \times S^1$ component, from which we get that there are exactly two and that M is a lens space.

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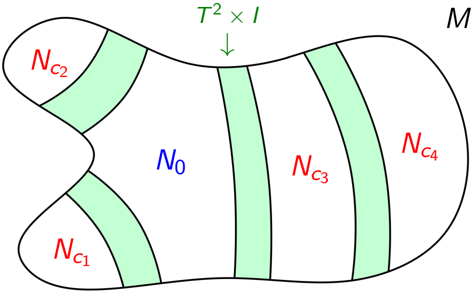
3. General case: f is non-constant.



- First, contact strong non-degeneracy + non-constant slope in each integrable domain is a C^1-dense property.

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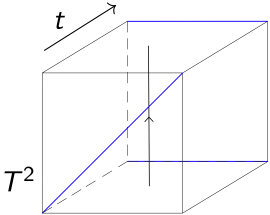
We end up with several $T^2 \times I$ regions, with non-constant slope, with sections to the flow near the boundary.

Theorem

Let X be a T^2-invariant flow on $T^2 \times I$ with a non-constant slope. Then given two families of curves Γ_0, Γ_1 such that $\Gamma_0 \subset T^2 \times \{0\}$ and $\Gamma_1 \subset T^2 \times \{1\}$ with $X \pitchfork \Gamma_i$, there exists a Birkhoff section S such that $S \cap T^2 \times \{0\} = \Gamma_0$ and $S \cap T^2 \times \{1\} = \Gamma_1$.

Main tool: Helix boxes

A closed orbit can be used to change the homology. Example:



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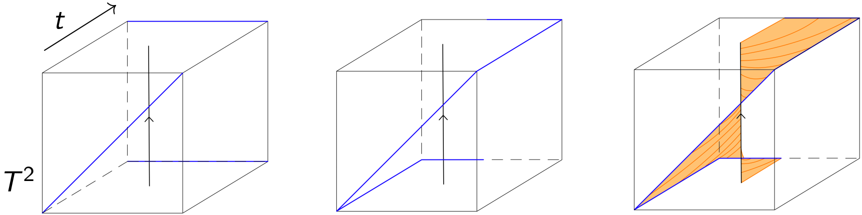
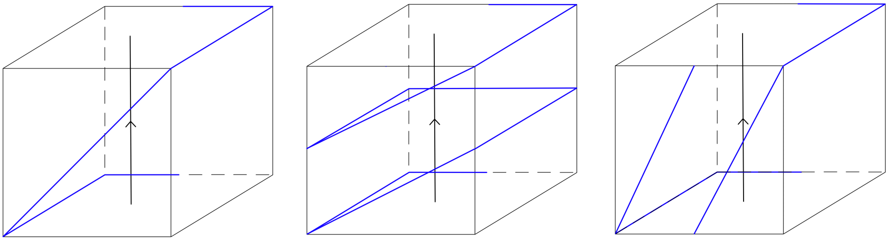
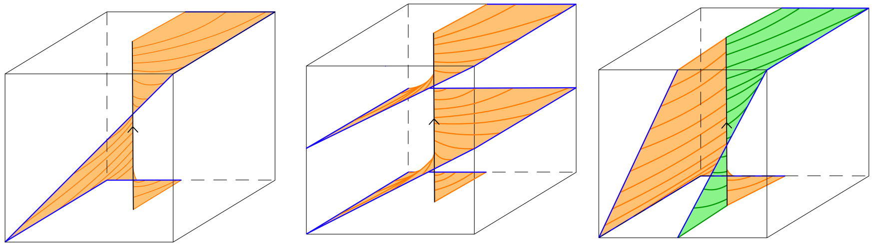


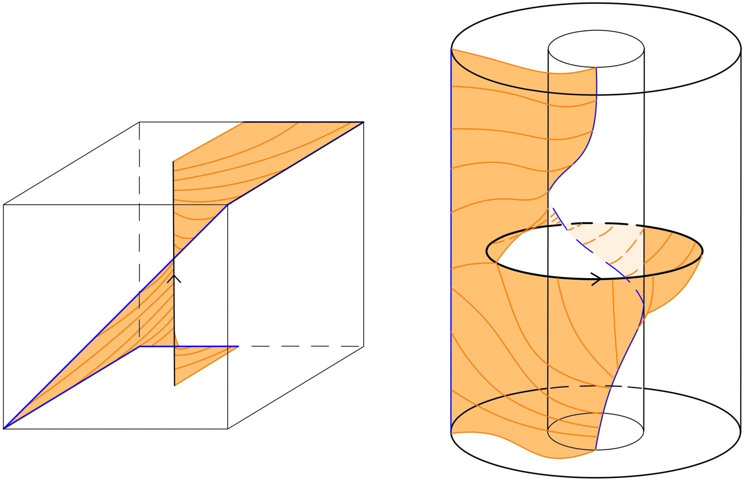
Boundary segments



Surfaces in cube



Smooth versions



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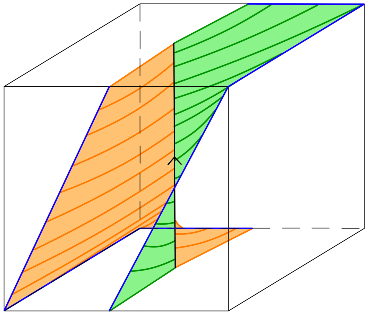

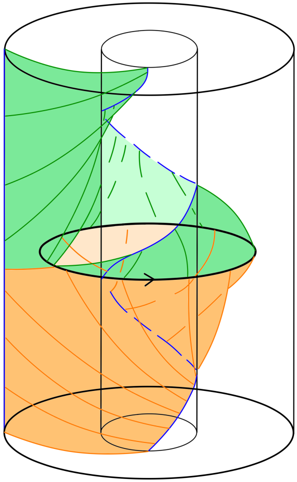

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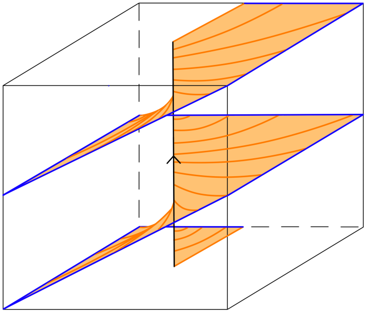

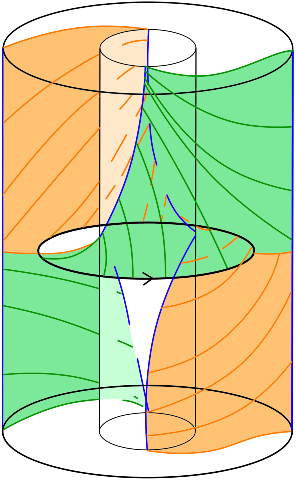

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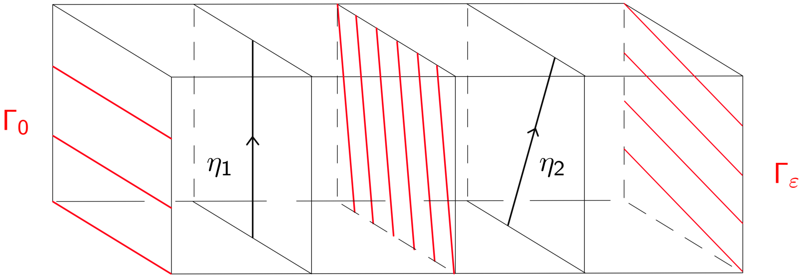

Choose generators η_1 and η_2 on two rational tori with close slope.

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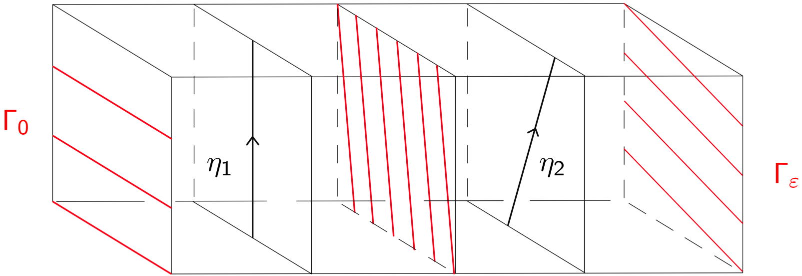

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$$[\Gamma_\varepsilon] = [\Gamma_0] + k_1[\eta_1] + k_2[\eta_2].$$

Key point: make sure that the intermediate section remains transverse before reaching γ_1.

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Question

Does every Reeb field defined by a SHS (perhaps non-aperiodic) admit a Birkhoff section?

Thanks for your attention