Braid stability and the Hofer metric

(Joint work with Matthias Meiwes)

1) How to associate a braid to a set of periodic orbits of a Hamiltonian diffeo.

Setup:

(Σ, ω) - compact surface endowed with symplectic form

ϕ - Hamiltonian diffeo
(if Σ has boundary $\phi = id$ near $\partial\Sigma$)

- $H: S^1 \times \Sigma \to \mathbb{R}$ a 1-periodic Hamiltonian generating ϕ

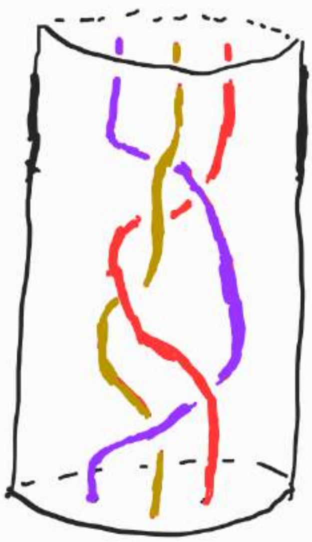
- $\phi = \phi_H^1$ where ϕ_H^t is the Hamiltonian flow of H on (Σ, ω).

$\boxed{S^1 = \mathbb{R}/\mathbb{Z}}$

- $P = \{p_1, \dots, p_m\}$ collection of distinct fixed points of ϕ.

- $\gamma_i = \{(t, \phi_H^t(p_i)) \mid t \in S^1\}$ $\boxed{\gamma_i \subset S^1 \times \Sigma}$

$$B_H(P) := \bigcup_{x_i \in P} \gamma_i$$



Hofer metric

$\phi, \phi' \in Ham(\Sigma, \omega) \longrightarrow$ group of Hamiltonianian diffeos of (Σ, ω)

$\mathcal{H}(\phi, \phi') \simeq$ normalized Hamiltonians generating $\phi^{-1} \circ \phi'$

- normalized : * closed $\Sigma \to \int_\Sigma H_t(x)\omega = 0 \quad \forall t \in S^1$

* $\partial\Sigma \neq \emptyset \to H(t)$ vanishes near $\partial\Sigma$

For $F : \Sigma \to \mathbb{R}$

$$\|F\| = \max F - \min F$$

$$d_H(\phi, \phi') = \inf_{H \in \mathcal{H}(\phi, \phi')} \int_0^1 \|H_t\| \, dt$$

Braid stability

Theorem: (A.-Meiwes)

Let $\phi \in Ham(\Sigma, \omega)$ and $\mathcal{P} = \{p_1, \dots, p_n\}$ be a finite collection of non-degenerate distinct fixed points of ϕ.

Then, $\exists\ \varepsilon > 0$ such that $\forall$ non-degenerate $\phi' \in Ham(\Sigma, \omega)$ with $d_H(\phi, \phi') < \varepsilon$ there is $\mathcal{P}' = \{p_1', \dots, p_n'\}$ distinct fixed points of ϕ' and H' generating ϕ' such that:

$$\mathcal{B}_H(\mathcal{P}) \cong \mathcal{B}_{H'}(\mathcal{P}')$$

freely isotopic as braids

Consequences

Theorem: (A. - Meiwes)

h_{top} is lower semi-continuous on $(\mathrm{Ham}(\Sigma,\omega), d_H)$.

Braid stability + approximate h_{top} by entropy of braids

Khanevsky observation:

- the set of entropy 0 Hamiltonian diffeomorphisms in $\mathrm{Ham}(\Sigma,\omega)$ is not dense in any d_H-open set.

Theorem (Khanevsky)

Autonomous in $\mathrm{Ham}(\Sigma,\omega)$ is not d_H-dense in "0-entropy".

there are "0-entropy" braids which are not autonomous.

Corollary: (Braid stability + CGG)

Çineli
Ginzburg
Gürel

The barcode entropy h_{bar} of CGG is lower semi-continuous in $(Ham(\Sigma,\omega), d_H)$.

Related results:

- Connery-Crigg (on braids)
  Some trivial braids are stable.

- Chor-Meiwes (on entropy)
  Given $N>0$, examples of arbitrarily large d_H-balls on which $h_{top} > N$.

Sketch of proof: (Assuming ϕ non-deg.)

+ Periodic orbits survive sufficiently small perturbations in d_H.
(Polterovich - Shelukhin, Usher, Usher-Zhang)

$P = \{p_1, \ldots, p_m\}$ distinct fixed points with the same action a, and no other fixed point of ϕ has action a.

$$\boxed{a := A_H(p_i)}$$

Let $\varepsilon > 0$ be such that

$$\mathrm{Spec}(\phi) \cap (a - 8\varepsilon, a + 8\varepsilon) = \{a\}$$

Consider $CF^{(a-\varepsilon, a+\varepsilon)}(H) = \mathbb{Z}_2$ - vector space generated by P

$$\cong \qquad\qquad " \bigoplus_m \mathbb{Z}_2$$

$$HF^{(a-\varepsilon, a+\varepsilon)}(H)$$

All orbits have the same action so the Floer differential d_H vanishes.

- $\phi' \in Ham(\Sigma, \omega)$ non-deg. and with $d_H(\phi, \phi') < \varepsilon$.

- From $d_H(\phi, \phi') < \varepsilon$ one constructs a normalized Hamiltonian H' generating ϕ' and homotopies

$$G : \mathbb{R} \times S^1 \times \Sigma \to \mathbb{R} \qquad \begin{array}{ll} G_s \equiv H & s \leq -1 \\ G_s \equiv H' & s \geq 1 \end{array}$$

$$\hat{G} : \mathbb{R} \times S^1 \times \Sigma \to \mathbb{R} \qquad \begin{array}{ll} \hat{G}_s \equiv H' & s \leq -1 \\ \hat{G}_s \equiv H & s \geq 1 \end{array}$$

such that we have continuation maps:

$$\Phi_G : CF^{(a-\varepsilon, a+\varepsilon)}(H) \longrightarrow CF^{(a-2\varepsilon, a+2\varepsilon)}(H')$$

$$\Phi_{\hat{G}} : CF^{(a-2\varepsilon, a+2\varepsilon)}(H') \longrightarrow \underline{CF^{(a-3\varepsilon, a+3\varepsilon)}(H)}$$

$$/\!/ \quad CF^{(a-\varepsilon, a+\varepsilon)}(H)$$

and

$$\Phi_{\hat{G}} \circ \Phi_G : CF^{(a-\varepsilon, a-\varepsilon)}(H) \longrightarrow CF^{(a-3\varepsilon, a+3\varepsilon)}(H)$$

$$\| \quad CF^{(a-\varepsilon, a+\varepsilon)}(H)$$

is chain-homotopic to the identity! $\boxed{\text{It's actually the identity, because } d_H = 0}$

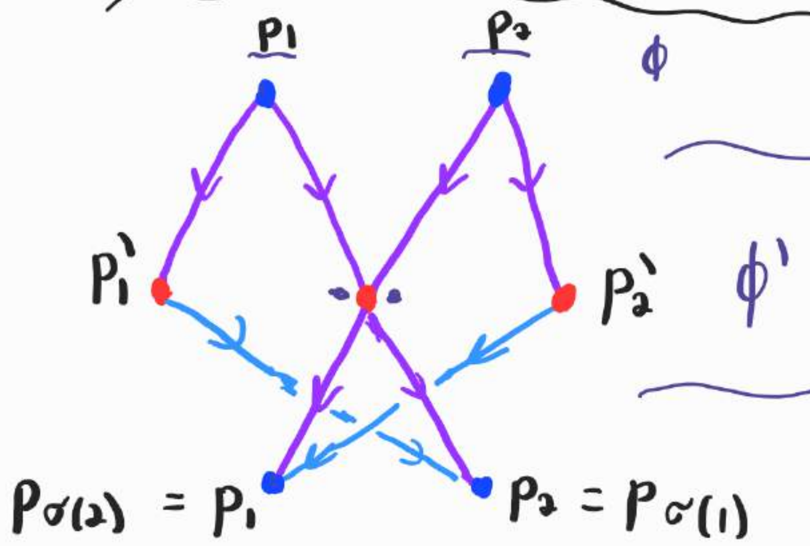
Conclusion: $\dim\left(CF^{(a-2\varepsilon,\, a+2\varepsilon)}(H')\right) \geqslant n$

+ Finding the right fixed points $P' = \{p_1', \dots, p_n'\}$ of ϕ'.

Combinatorial lemma:

There exist $\{p_1', \dots, p_n'\}$ distinct fixed points of H' and a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ s.t.:

- there is a G-Floer cylinder U_i from p_i to p_i',
- there is a $\hat{G}$-Floer cylinder V_i from p_i' to $p_{\sigma(i)}$.



$$\begin{tikzcd} CF^{(a-\varepsilon,\, a+\varepsilon)}(H) \arrow[d, "G"] \\ CF^{(a-2\varepsilon,\, a+2\varepsilon)}(H') \arrow[d, "\hat{G}"] \\ CF^{(a-\varepsilon,\, a+\varepsilon)}(H) \end{tikzcd}$$

Lift the u_i and v_i to holomorphic cylinders $\tilde{u}_i$ and $\tilde{v}_i$ (for alm. complex struct. J_G and $J_{\hat{G}}$) in $\mathbb{R} \times S^1 \times \Sigma$ (Gromov trick).

Compactify $\tilde{u}_i$ and $\tilde{v}_i$ to $\overline{u}_i$ and $\overline{v}_i$ in $\overline{\mathbb{R}} \times S^1 \times \Sigma$.

Some facts:

- $\overline{u}_i$ and $\overline{v}_i$ are embedded cylinders,
- $\overline{u}_i$ and $\overline{u}_j$ can only have interior intersection points if $i \neq j$ (same for $\overline{v}_i$ and $\overline{v}_j$)

$\gamma_{\sigma(i)} = \{(t, \phi_H^t(p_{\sigma(i)})) \mid t \in S^1\}$

$\overline{V}_i$

$\gamma'_i = \{(t, \phi_{H'}^t(p'_i)) \mid t \in S^1\}$

$\overline{U}_i$

$\gamma_i = \{(t, \phi_H^t(p_i)) \mid t \in S^1\}$

Positivity of intersections tells us that $\overline{V}_i$'s intersect positively.

$\text{int}(\overline{V}_i \cap \overline{V}_j) \geq 0$

$\boxed{\overline{U}_i(s,\cdot) \text{ is a knot isotopy}}$

$\cap \{s\} \times S^1 \times \Sigma$

+ Show that $\overline{U}_i$ and $\overline{U}_j$ do not intersect for $i \neq j$.

(Easy case $\Sigma = \mathbb{D}$)

Pilling up cylinders.

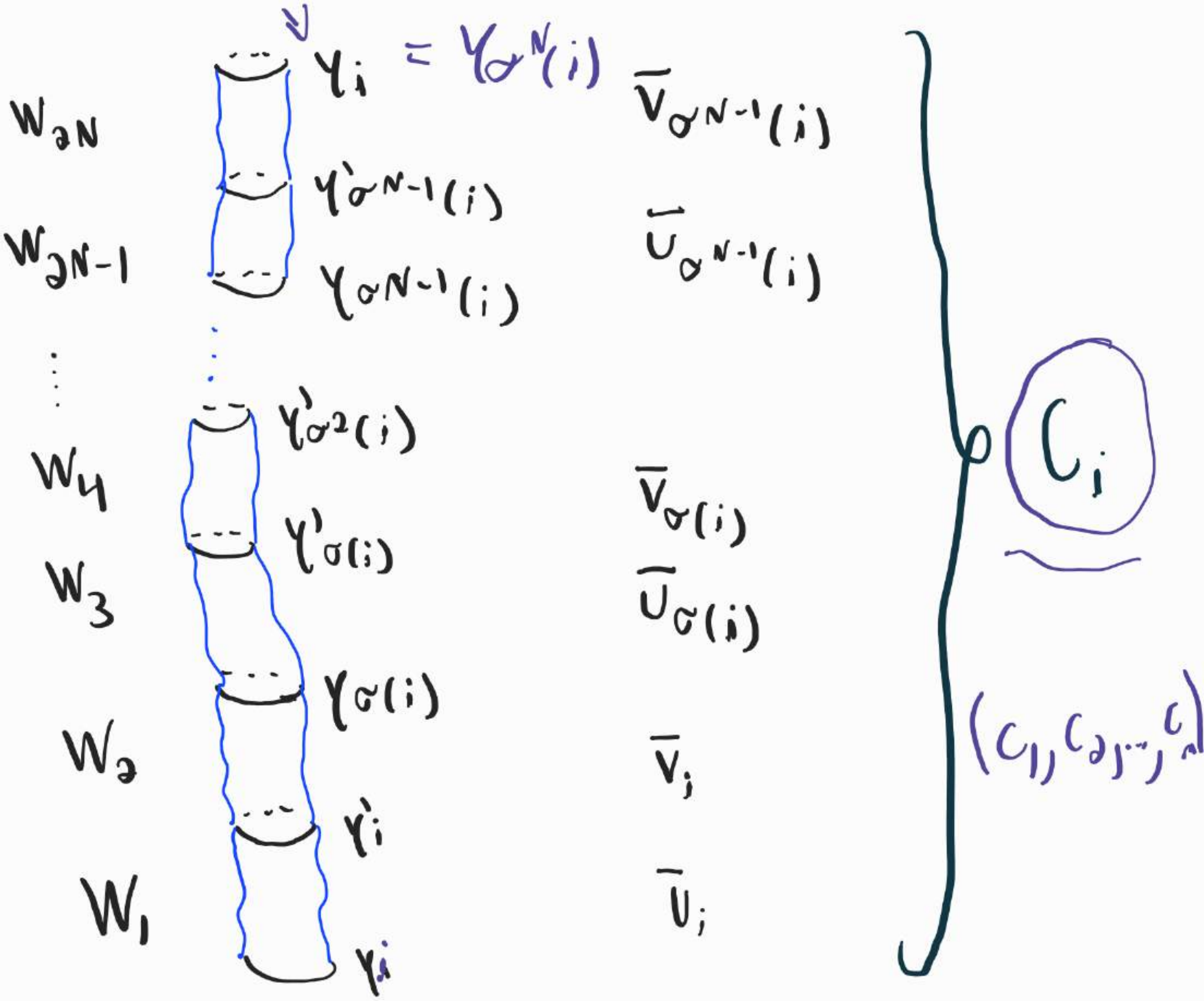
$N :=$ order of σ

concatenate $2N$ copies of $W := \overline{\mathbb{R}} \times S^1 \times \Sigma$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \cong$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad I \times S^1 \times \Sigma$

$(W_1, W_2, \dots, W_m)$



- C_i is homotopic with ends fixed to the straight cylinder C_i^0 over γ_i (there's no topology ID)

- For $i \neq j$,

$$\mathrm{int}(C_i, C_j) > 0 \quad \text{if} \quad \overline{U}_i \cap \overline{U}_j \neq \emptyset$$

- But

$$\mathrm{int}(C_i, C_j) \overset{\substack{\text{invariance}\\ \downarrow}}{=\!=} \mathrm{int}(C_i^0, C_j^0) \underset{\substack{\downarrow\\ \text{obvious}}}{=} 0$$

- Conclusion:

$$\overline{U}_i \cap \overline{U}_j = \emptyset \qquad \forall \quad i \neq j$$

- The cylinders $\overline{U}_1, \overline{U}_2, \dots, \overline{U}_n$ give the braid isotopy. $\square$

Comments:

- the braid isotopy is given by the Floer cylinders,
- the orbits y_i and y_i' are not close in any metric sense.

Braids and surface dynamics

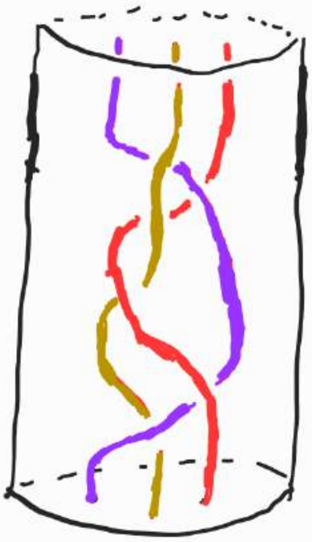
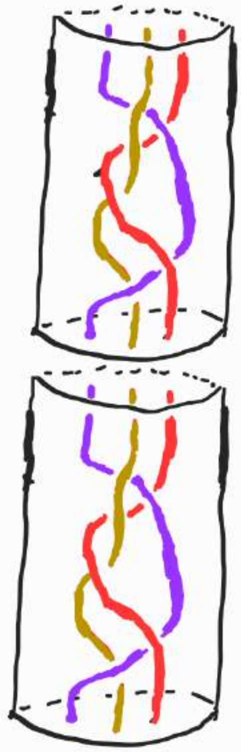
Let $Q = \{q_1, \ldots, q_m\}$ be a finite set of points on Σ.

$MG(\Sigma, Q)$ = " isotopy classes of diffeos of Σ that preserve the set Q"

$\boxed{\text{Mapping class group}}$

If $\partial\Sigma \neq \emptyset$ we assume that the diffeos preserve each boundary component.

Let $Br(Q)$ be the group of braids based on Q.

$\beta =$  $\longrightarrow$ β^2 

Birman:

There is a homomorphism

$$\Psi : Br(Q) \longrightarrow MG(\Sigma, Q)$$

To get a diffeo from a braid β push points in Q along the braid, and extend it to a diffeo of Σ.

"$\Psi(\beta)$ is sometimes the braid type of β by dynamicists"

One can use dynamics to study braids, and the other way around.

For each element $[f]$ of $MG(\Sigma, Q)$ one can look at:

$$\min_{g \in [f]} \left(h_{top}(g) \right) =: h_{top}([f])$$

this is the h_{top} of Thurston-Nielsen representative of $[f]$

The topological entropy of a braid β is

$$h_{top}(\beta) := h_{top}(\Psi(\beta))$$

Theorem (A. – Meiwes)

Let ϕ be a C^2-diffeo of a compact surface. Then:

$$h_{top}(\phi) = \sup_{\substack{\beta \text{ is realised} \\ \text{as a periodic} \\ \text{orbit of } \phi}} h_{top}(\beta)$$

Hall: in special cases

Proof:

Mixture of Katok's techniques (NUH-dynamics) and an idea of Franks – Handel.

(Nielsen theory)

Dynamics carried by a braid is stable w.r.t. d_H-perturbations.

h_{top}, Boyland's partial order, etc...

$\square$

Future goals:

- Reeb flows in dim 3
  (d_H is substituted by the C^0-distance on the space of contact forms)

  Substitute braids by transverse links.

  Transverse links can force h_{top}

  $\rightsquigarrow$ by work of A. – Pirnapasov using CH on the complement (Momin)

Theorem (Meiwes)

$\to$ h_{top} of a 3-dimensional C^∞-generic Reeb flow can be recovered from the CH on complements of links of periodic orbits.

Analogous of braid stability for C^0-perturbations of 3-d contact forms: joint project with Meiwes, LUCAS DAHINDEN and ABROR PIRNAPASOV.

– Higher dimensions.

Almost nothing is known.

Some symplectic manifolds have d_H-open sets on which $h_{top} > c > 0$.

We don't believe h_{top} is lower semi-continuous w.r.t. d_H in $\dim \geq 4$.

But:

$\mathrm{Ent}_0(M, \omega) :=$ "Ham diffeos with $h_{top} = 0$"

Conjecture:

$\mathrm{Ent}_0(M, \omega)$ is not dense in any open d_H-ball.

THANKS!