Arnold conjecture over integers

(Joint with Guangbo Xu)

I. Statement of the result

II. Proof outline

III. Selected technical details

Based on arXiv: 2209.08599

I.

(M, ω) closed, symplectic

$H_t : S^1 \times M \to \mathbb{R}$ 1-periodic C^∞-Hamiltonian

$dH_t = \omega(X_{H_t}, -)$ Hamiltonian vector field

$\varphi^1_{H_t} \Rightarrow$ time-1 flow of X_{H_t}

$\forall x \in \mathrm{Fix}(\varphi^1_{H_t})$, $\det(d\varphi^1_{H_t}(x) - Id) \neq 0$

all fixed points are non-degenerate

Goal. Lower bound for $\# \mathrm{Fix}(\varphi^1_{H_t})$

$N \Rightarrow$ minimal Chern number

i.e., the maximal $N \in \mathbb{Z}_{>0} \cup \{\infty\}$ s.t. $\frac{1}{N} c_1(M,\omega) \in H^2(M;\mathbb{Z})$

Consider: $H_*^{(N)}(M;\mathbb{Z}) := \bigoplus_{\substack{i \equiv * \\ \text{mod} 2N}} H_i(M;\mathbb{Z})$

(integral homology of M with reduced $\mathbb{Z}/2N$-grading)

finitely generated module over $\mathbb{Z}$

$\mathbb{Z}$ is P.I.D. $\Rightarrow \exists\, b \in \mathbb{Z}_{\geq 0}$, $a_1 | \cdots | a_k$ positive integers, s.t.

$H_*^{(N)}(M;\mathbb{Z}) \cong \mathbb{Z}^{\oplus b} \oplus \mathbb{Z}/a_1 \oplus \cdots \oplus \mathbb{Z}/a_k$

invariant factors

Define. $b_*^{(N)}(M) := b + 2k$ "integral Betti number"

Thm. (B-Xu) If $H_t : S^1 \times M \to \mathbb{R}$ non-degenerate,

then $\# \mathrm{Fix}(\varphi_{H_t}^1) \geq \sum_{i \in \mathbb{Z}/2N} b_i^{(N)}(M)$

Rmk. By universal coefficient theorem, we recover

- Fukaya-Ono, Liu-Tian (rational Betti numbers)
- Abouzaid-Blumberg ($\mathbb{F}_p$ Betti numbers)

Stronger: torsion components of different characteristics lying in different $H_*^{(N)}$

History: Conley-Zehnder, Floer, Ono, Hofer-Salamon ...

Discussion

- If $N=1$, $\exists$ $\mathbb{Z}/2$ in H_7, $\mathbb{Z}/3$ in H_9,

  then our result is only as strong as Abouzaid-Blumberg's result.

- Is it possible to obtain stable Morse number as a lower bound?

  $F: M \times \mathbb{R}^k$ Morse + quadratic @ ∞

  minimal # of critical points

  Bifurcation method.

II. Outline : Floer homology over $\mathbb{Z}$.

$\Lambda_{\mathbb{Z}}$ $\Rightarrow$ Novikov ring with integer coefficients

Thm (B-Xu)

(a). For (M,ω), $H_t: S^1 \times M \to \mathbb{R}$ non-degenerate,

$\exists$ chain complex $(CF_*(M;H_t), d) \rightsquigarrow HF_*(H_t;\Lambda_{\mathbb{Z}})$

$\mathbb{Z}/2N$-graded, freely generated over $\Lambda_{\mathbb{Z}}$ by $Fix(\varphi^1_{H_t})$

(b) (f,g) Morse-Smale, $CM_*(f;\Lambda_{\mathbb{Z}}) := CM_*(f;\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}$

$\exists$: $PSS: CM_*(f;\Lambda_{\mathbb{Z}}) \to CF_*(M;H_t)$
$SSP: CF_*(M;H_t) \to CM_*(f;\Lambda_{\mathbb{Z}})$ chain maps

s.t. $SSP \circ PSS = Id + O(T^{\delta})$, for some $\delta > 0$.

Cor. $PSS : H_*(M; \Lambda_{\mathbb{Z}}) \to HF_*(H_t; \Lambda_{\mathbb{Z}})$ is an injection

$\rightsquigarrow$ Arnold follows by elementary algebra.

Rmks. (1) Haven't established the independence of $HF_*(H_t; \Lambda_{\mathbb{Z}})$ on H_t, but it's true in principle

(2) No use of Floer homotopy theory, but conceptually related.

(3) Work in more general settings, e.g.,

Lagrangian Floer theory over $\mathbb{Z}$ for rel. Spin Lagrangians.

(in progress)

Ingredients.

(A). A new perturbation method realizing a proposal of Fukaya–Ono

- producing integral pseudo-cycles from normally complex derived orbifolds $\Rightarrow$ cf. earlier work w/. Xu
- Floer: multiplicativity of transversality $\Rightarrow$ new

$$\partial \bar{\mathcal{M}}(p,q) = \bigcup_r \bar{\mathcal{M}}(p,r) \times \bar{\mathcal{M}}(r,q)$$

(B) Regularization of moduli spaces (developing Abouzaid–McLean–Smith)

- global charts for $\bar{\mathcal{M}}(p,q)$ (see also recent arXiv post by S. Rezchikov)
- Smoothing manifolds with corners.

$\underline{\text{III}}$.

Normally complex polynomial perturbation

derived orbifold chart: $(D, \mathcal{E}, S)$

orbibundle

continuous section

smooth orbifold

compact: $S^{-1}(0)$ compact

normally complex: $\forall x \in D$, let $(\mathcal{U}, \Gamma, \psi)$ be an orbifold chart

$T_x\mathcal{U} \circlearrowleft \Gamma = (T_x\mathcal{U})^\Gamma \oplus (T_x\mathcal{U})^\perp$

then require $(T_x\mathcal{U})^\perp$ is a complex Γ-representation

Similarly, let $\begin{matrix} E \\ \downarrow \\ \mathcal{U} \end{matrix}$ be an orbibundle chart

$E_x = E_x^\Gamma \oplus E_x^\perp$, $E_x^\perp$ is complex

Thm (B-Xu). $(D, \mathcal{E}, S)$ compact normally complex derived orbifold chart. Then $\forall\, \varepsilon > 0$, $\exists$ $\begin{matrix} \mathcal{E} \\ \downarrow \\ D \end{matrix} \circlearrowleft S'$ strongly transverse normally complex polynomial

s.t. $\|S' - S\|_{C^0} < \varepsilon$, $(S')^{-1}(0)$ is a free pseudo-cycle.

Any two such pseudo-cycles are cobordant.

$\Longrightarrow$ produces integral homology class in an obvious way.

- For $(D_1, \mathcal{E}_1, S_1)$, $(D_2, \mathcal{E}_2, S_2)$, if S_1', S_2' are — perturbations, so is their product $\begin{matrix} \pi_1^*\mathcal{E}_1 \oplus \pi_2^*\mathcal{E}_2 \\ \downarrow \\ D_1 \times D_2 \end{matrix} \circlearrowleft \pi_1^* S_1' \oplus \pi_2^* S_2'$

(Fukaya-Ono-Parker perturbation)

Claim. "All" moduli space of pseudo-holomorphic curves are normally complex.

(need to be careful in the SFT setting)

Morally. (in genus 0)

- nontrivial isotropy is caused by multiply-covered spheres
- for a stable map $u: \Sigma \to M$, the non Γ_u-invariant part of deformation complex

$$D_u : \Omega^0(\Sigma, u^* TM) \circlearrowleft \Gamma_u \to \Omega^{0,1}(\Sigma, u^* TM) \circlearrowleft \Gamma_u$$

"localizes" along the spherical components.

- $D_u|_{\text{spherical components}}$ is homotopic to a complex linear operator

$\rightsquigarrow \ker^{\perp} - \operatorname{coker}^{\perp}$ is a virtual complex vector space

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As a result, once the moduli spaces are regularized to have smooth structures, the FOP perturbation method can define integer-valued counts in very general settings.

$\rightsquigarrow$ may be useful for problems using positive characteristics.