Symplectic instanton homology of knots and links

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Outline

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1. Introduction
2. Construction
3. Principal results
4. Ongoing investigation

Motivations

- Singular instanton knot homology (Kronheimer and Mrowka 2011b; Kronheimer and Mrowka 2011a).

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- Atiyah-Floer conjecture (Atiyah 1988).
- 3-mfld symplectic instanton homology $\mathrm{SI}(Y)$ via traceless character varieties (Horton 2017).

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Morse homology on some associated infinite-dimensional "configuration space" $\mathcal{C}$.

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   - $\mathcal{C} =$ irred. principal G-connections mod gauge, $G =$ an appropriate Lie group.
   - Example: singular instanton knot homology (Kronheimer and Mrowka 2011b).

Here we consider just $G = \mathrm{SU}(2)$.

Atiyah-Floer relationship

Holonomy correspondence:

$$\{\text{instanton-Floer generators}\} \quad \leftrightarrow \quad \{\mathrm{SU}(2)\text{-characters}\}$$

$$\frac{\{\text{flat } G-\text{connections}\}}{\text{gauge}} \quad \leftrightarrow \quad \frac{\{\text{representations } \rho \in \mathrm{Hom}(\pi_1, G)\}}{\text{conj.}}.$$

Atiyah-Floer conjecture (Atiyah 1988)

Given Heegaard splitting $Y = U_0 \cup_\Sigma U_0$,

$$I_*(Y) \cong \mathrm{HF}(\mathcal{R}(\Sigma); \mathcal{R}(U_0), \mathcal{R}(U_1)).$$

More generally: Instanton and Lagrangian Floer homologies come in isomorphic pairs.

—Based on the same reasoning which lead to the reformulation (Taubes 1990) of the Casson invariant.

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  - $Y =$ connected closed oriented 3-mfld;
  - $K \subset Y$ a knot or link.
- Extend the **quilted Floer theory** (Wehrheim and C. T. Woodward 2010) for tangles (Wehrheim and C. Woodward 2016).

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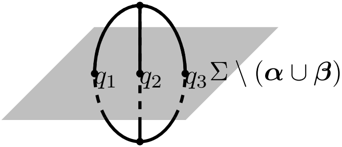

Figure: Theta graph in a neighborhood of $Y \setminus K$.

Lagrangians from Heegaard diagrams

Let $M = \mathscr{R}_{g,2k+3}$ = **traceless** $\mathrm{SU}(2)$-**charachter variety** of $(\Sigma_g, \mathbf{w} \cup \mathbf{z} \cup \mathbf{q})$.

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Theorem (Cazassus, Herald, and Kirk 2021; Alekseev, Malkin, and Meinrenken 1998)

$L_\alpha, L_\beta \subset M$ are embedded Lagrangians.

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$\therefore L_\alpha, L_\beta$ have **min. Maslov number** $N_L = 2c_1(\mathscr{R}_{g,n}) = 2(1) = 2$.

Establishing Floer homology

Theorem ((Oh 2015, Cor. 16.4.8))

Let (L_0, L_1) *be a pair of connected, compact monotone Lagrangian submanifolds in a symplectic manifold* (M, ω) *having minimal Maslov number* $N_L \geq 2$. *For* $i = 0, 1,$ *let* a_i *be the positive generator of the group* $\{\omega(\beta) \mid \beta \in \pi_2(M, L_i)\} \subset \mathbb{Z}$, *and let* $\Phi(L_i)$ *be the* ***disk number*** *of* L_i. *Then* $\partial \circ \partial = 0$ *if and only if* $\Phi(L_0) = \Phi(L_1)$ *and* $a_0 = a_1$.

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Proof sketch for Theorem A.

1. L_α, L_β are the traceless character varieties of the handlebodies determined by $\mathcal{H}_k$ (relative to K, θ).
2. A diffeomorphism of these handlebodies induces a symplectomorphism of $\mathscr{R}_{g,n}$ which relates L_α, L_β.
3. $\Phi(L_i)$ and a_i are preserved under symplectormorphism, so the theorem above applies.

Topological invariance

Theorem (B)

Let $\mathcal{H}, \mathcal{H}'$ be two multi-pointed Heegaard diagrams compatible with (Y, K). Then

$$\mathrm{SI}(\mathcal{H}) \cong_{\mathit{can.}} \mathrm{SI}(\mathcal{H}'). \tag{1}$$

We define $\mathrm{SI}(Y, K) :=$ isom. class of $\mathrm{SI}(\mathcal{H})$ for any any compatible diagram $\mathcal{H}$.

Topological invariance

Proposition ((Manolescu and P. S. Ozsváth 2010))

Any two compatible $\mathcal{H}$ and $\mathcal{H}'$ are related by a finite sequence of the following multi-pointed Heegaard moves:

1. *isotopies,*
2. *handleslides,*
3. *index-one/two (de)stabilization (changing genus g),*
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Proof sketch for Theorem B.

- (1) & (2) leave π_1 of the handlebodies in the Heegaard splitting, hence L_α and L_β, unchanged.
- Apply Wehrheim & Woodward's *embedded composition thm.* to quilted Floer homologies arising from (3) & (4).

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- Skein exact sequence.

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