Lagrangian Hofer metric and barcodes

Patricia Dietzsch

ETH Zürich

Zoominar

Table of Contents

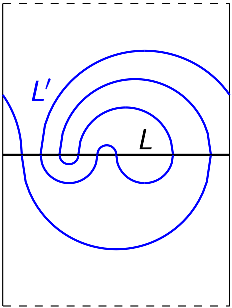
1. Setting: $L, L' \subset$ cylinder
2. Persistence Floer homology and barcodes:
   $\beta_k(L, L') \leq \cdots \leq \beta_1(L, L') \leq \gamma(L, L') \leq d_H(L, L')$
3. Main result: $d_H(L, L') \leq \sum 2^j \beta_j(L, L') + \gamma(L, L')$
4. Idea of proof: remove the smallest bar

Table of Contents

Setting

Consider

- (Σ, ω): Unit cotangent bundle of S^1
- $L, L' \subset \Sigma$: Lagrangian submanifolds,
  Hamiltonian isotopic to the zero-section,
  intersecting transversely.



Lagrangian Hofer metric

Consider

- (Σ, ω): Unit cotangent bundle of S^1
- $L, L' \subset \Sigma$: Lagrangian submanifolds,
  Hamiltonian isotopic to the zero-section,
  intersecting transversely.

Lagrangian Hofer metric:

$$d_H(L, L') = \inf \left\{ \int_0^1 \max_{x \in \Sigma} H_t(x) - \min_{x \in \Sigma} H_t(x) \, \mathrm{d}t \;\middle|\; \begin{array}{c} H \in C_c^\infty(\mathbb{R} \times \Sigma), \\ \psi_1^H(L) = L' \end{array} \right\}$$

Table of Contents

Barcode associated to a pair of Lagrangians

Barcode associated to a pair of Lagrangians

Goal:
An upper bound of $d_H(L, L')$ in terms of invariants extracted from $\mathcal{B}(L, L')$.

Floer complex

Consider

- (Σ, ω): Unit cotangent bundle of S^1
- $L, L' \subset \Sigma$: Lagrangian submanifolds,
  Hamiltonian isotopic to the zero-section,
  intersecting transversely.

Floer complex:

$$\mathrm{CF}(L, L') = \bigoplus_{q \in L \cap L'} \mathbb{Z}_2 q$$

Floer complex

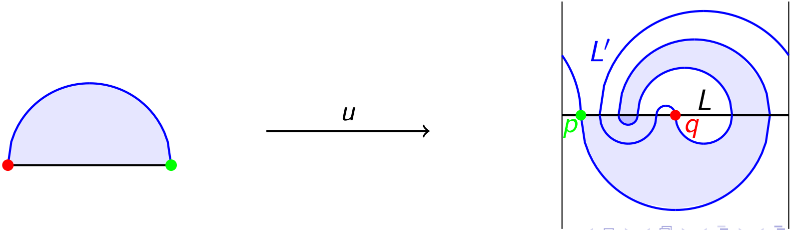
Consider

- (Σ, ω): Unit cotangent bundle of S^1
- $L, L' \subset \Sigma$: Lagrangian submanifolds,
  Hamiltonian isotopic to the zero-section,
  intersecting transversely.

Floer complex:

$$\mathrm{CF}(L, L') = \bigoplus_{q \in L \cap L'} \mathbb{Z}_2 q$$

Differential counts smooth orientation-preserving immersions:



Filtration

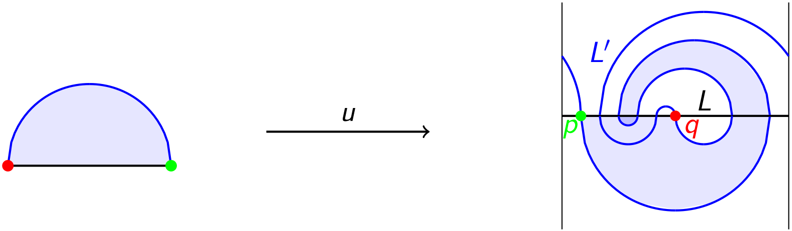
Action functional:

$$\mathcal{A}\colon L \cap L' \longrightarrow \mathbb{R}$$

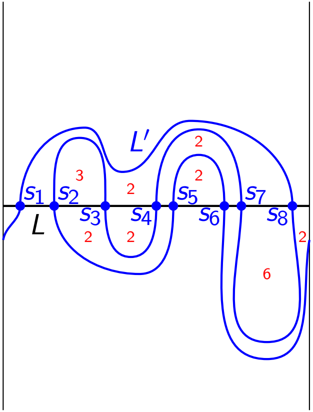
satisfying

$$\mathcal{A}(q) - \mathcal{A}(p) = \int u^*\omega,$$

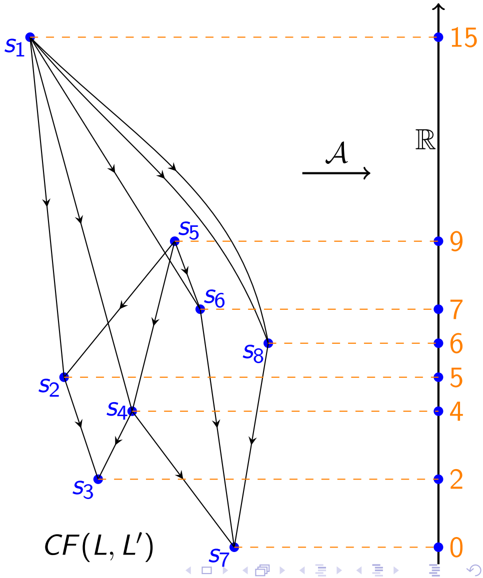
whenever



Example: filtered Floer complex



(L, L')



CF(L, L')

Barcodes

Barcode: finite multiset $\mathcal{B} = \{I_j\}_{j=1}^n$ of intervals I_j of two possible types:

- Finite bars: $I_j = [a_j, b_j)$, where $a_j < b_j \in \mathbb{R}$
- Infinite bars: $I_j = [c_j, \infty)$, where $c_j \in \mathbb{R}$

Barcodes

Barcode: finite multiset $\mathcal{B} = \{I_j\}_{j=1}^n$ of intervals I_j of two possible types:

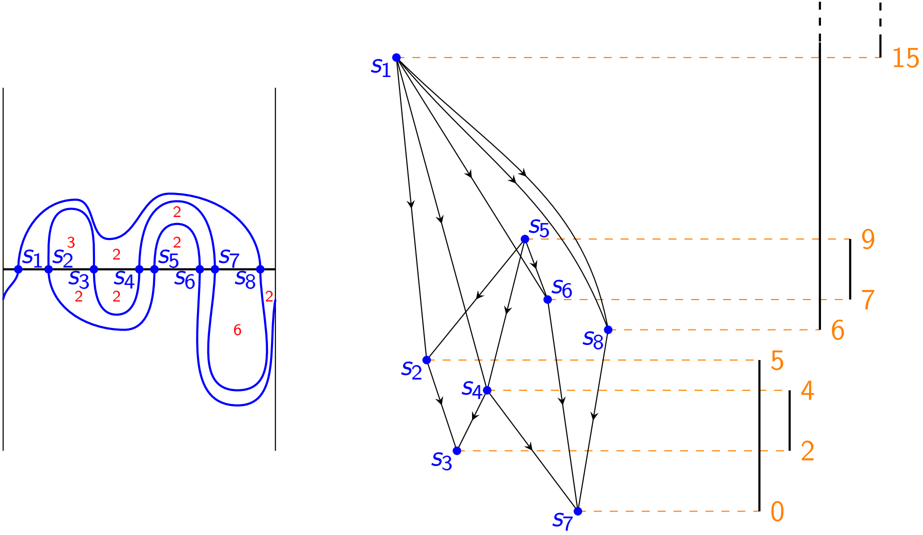
- Finite bars: $I_j = [a_j, b_j)$, where $a_j < b_j \in \mathbb{R}$
- Infinite bars: $I_j = [c_j, \infty)$, where $c_j \in \mathbb{R}$

Assume $\mathcal{A}(q)$ are all distinct. Some properties of $\mathcal{B}(L, L')$:

- $\# \{\text{infinite bars}\} = \mathrm{rank}\, H_*(L) = 2$
- $\{\text{Endpoints of the bars}\} = \mathcal{A}(L \cap L')$
- For $q \in L \cap L'$,

$$\mathcal{A}(q) \text{ is a lower end of a bar}$$
$$\Leftrightarrow$$
$$\partial(q) \text{ is a boundary in } \mathrm{CF}^{<\mathcal{A}(q)}(L, L')$$

Example: $\mathcal{B}(L, L')$



(L, L') $CF(L, L')$ $\mathcal{B}(L, L')$

Invariants

Spectral metric:

$$\gamma(L, L') = \text{the difference of the endpoints of the two infinite bars}$$

Lengths of the finite bars:

$$\beta_1(L, L') \geq \beta_2(L, L') \geq \cdots \geq \beta_k(L, L')$$

Stability

Spectral metric:

$$\gamma(L, L') = \text{ the difference of the endpoints of the two infinite bars}$$

Lengths of the finite bars:

$$\beta_1(L, L') \geq \beta_2(L, L') \geq \cdots \geq \beta_k(L, L')$$

Stability properties [Polterovich-Shelukhin 2014]:

- $\beta_1(L, L') \leq \gamma(L, L') \leq d_H(L, L')$ [Kislev-Shelukhin 2018]
- $|\gamma(L, L') - \gamma(L, L'')| \leq d_H(L', L'')$
- $|\beta_j(L, L') - \beta_j(L, L'')| \leq d_H(L', L'')$ for all j

Table of Contents

Main result

L, L': Lagrangians, Hamiltonian isotopic to the zero-section in the cylinder

Theorem (D. 2022)

Suppose that L, L' intersect transversely in $2n$ points. Then

$$d_H(L, L') \leq \sum_{j=1}^{n-1} 2^j \beta_j(L, L') + \gamma(L, L').$$

Corollary (D. 2022)

For any L, L' as above

$$d_H(L, L') \leq 2^n \gamma(L, L').$$

Some consequences

$$d_H(L, L') \leq 2^n \gamma(L, L')$$

Let $\{L_k\}_{k\in\mathbb{N}}$ be a sequence of Lagrangians Hamiltonian isotopic to the zero-section L_0 satisfying $L_k \pitchfork L_0$.

Unbounded sequences [Khanevsky 2009]

$$d_H(L_0, L_k) \xrightarrow{k\to\infty} \infty \qquad \Longrightarrow \qquad \#(L_0 \cap L_k) \xrightarrow{k\to\infty} \infty.$$

because γ is bounded [Shelukhin 2018].

Some consequences

$$d_H(L, L') \leq 2^n \gamma(L, L')$$

Let $\{L_k\}_{k\in\mathbb{N}}$ be a sequence of Lagrangians Hamiltonian isotopic to the zero-section L_0 satisfying $L_k \pitchfork L_0$.

Unbounded sequences [Khanevsky 2009]

$$d_H(L_0, L_k) \xrightarrow{k\to\infty} \infty \qquad \Longrightarrow \qquad \#(L_0 \cap L_k) \xrightarrow{k\to\infty} \infty.$$

because γ is bounded [Shelukhin 2018].

Convergent sequences

Suppose the sequence $\#(L_k \cap L_0)$ is bounded. Then

$$L_k \xrightarrow{C^0} L_0 \qquad \Longrightarrow \qquad L_k \xrightarrow{d_H} L_0.$$

because γ is C^0-continuous [Buhovsky-Humilière-Seyfaddini 2019].

Table of Contents

Idea of proof: Step 1

Given: L, L' intersecting transversely in $2n$ points

$n = 1$: direct calucation

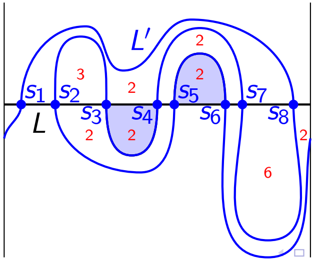
Assume: $n \geq 2$

1) Let $[\mathcal{A}(\bar{p}), \mathcal{A}(\bar{q}))$ be a smallest finite bar in $\mathcal{B}(L, L')$.

Proposition

There is a strip from $\bar{q}$ and $\bar{p}$. It has minimal area.

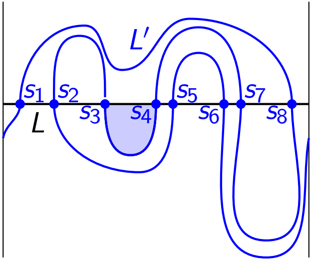
The area of the strip is $\beta_{n-1}(L, L')$.

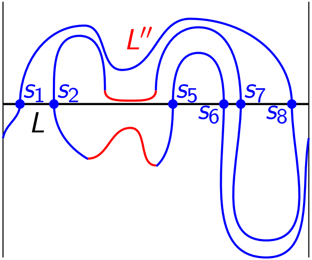


Idea of proof: Step 2

Given: L, L' intersecting transversely in $2n$ points, $n \geq 2$

1) There is minimal strip from $\bar{q}$ to $\bar{p}$ of area $\beta_{n-1}(L, L')$.
2) Remove the intersection points $\bar{p}, \bar{q}$: construct L'' such that
   - $d_H(L', L'') \leq \beta_{n-1}(L, L')$ (up to ϵ)
   - $L'' \pitchfork L$ and $\#(L'' \cap L) = 2(n-1)$
     [Khanevsky 2009, *deletion of a leaf*]





Idea of proof: Step 3

1),2) There is L'' with $\#(L'' \cap L) = 2(n-1)$ and $d_H(L', L'') \leq \beta_{n-1}(L, L')$.

3) Stability implies

$$|\beta_j(L, L') - \beta_j(L, L'')| \leq d_H(L', L'') \leq \beta_{n-1}(L, L')$$

for all $1 \leq j \leq n-2$ and

$$|\gamma(L, L') - \gamma(L, L'')| \leq d_H(L', L'') \leq \beta_{n-1}(L, L').$$

Idea of proof: Last Step

Apply induction hypothesis to (L, L''):

$$\begin{aligned} d_H(L, L') &\leq d_H(L, L'') + d_H(L'', L') \\ &\leq \left(\sum_{j=1}^{n-2} 2^j \beta_j(L, L'') + \gamma(L, L'') \right) + \beta_{n-1}(L, L') \\ &\leq \sum_{j=1}^{n-2} 2^j \left(\beta_j(L, L') + \beta_{n-1}(L, L') \right) + \gamma(L, L') + 2\beta_{n-1}(L, L') \\ &= \sum_{j=1}^{n-2} 2^j \beta_j(L, L') + \left(\sum_{j=1}^{n-2} 2^j + 2 \right) \beta_{n-1}(L, L') + \gamma(L, L') \\ &= \sum_{j=1}^{n-1} 2^j \beta_j(L, L') + \gamma(L, L') \end{aligned}$$

Thank you!

More properties of $\mathcal{B}(L, L')$

Barcode: finite multiset $\mathcal{B} = \{I_j\}_{j=1}^n$ of intervals I_j of two possible types:

- Finite bars: $I_j = [a_j, b_j)$, where $a_j < b_j \in \mathbb{R}$
- Infinite bars: $I_j = [c_j, \infty)$, where $c_j \in \mathbb{R}$

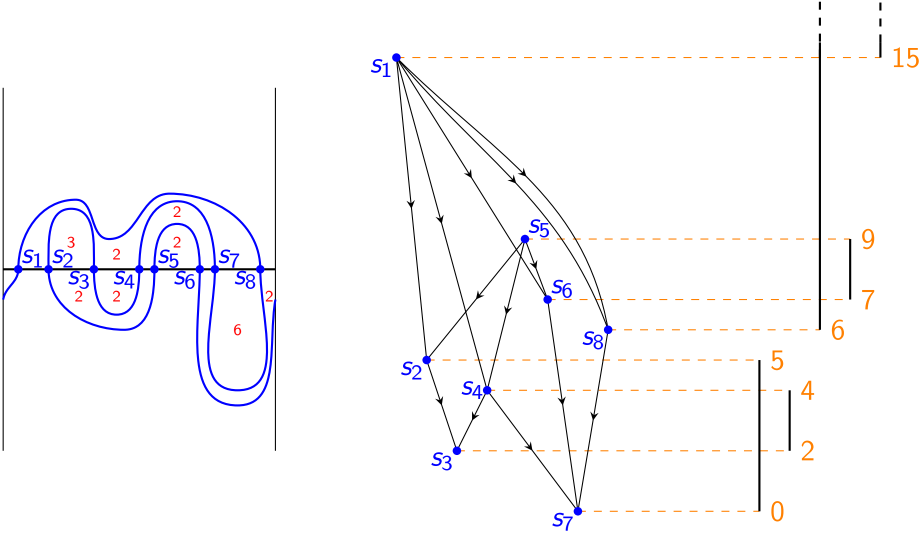
Assume $\mathcal{A}(q)$ are all distinct. Some properties of $\mathcal{B}(L, L')$:

- $\#\{\text{infinite bars}\} = \operatorname{rank} H_*(L) = 2$
- $\{\text{Endpoints of the bars}\} = \mathcal{A}(L \cap L')$
- For $q \in L \cap L'$,

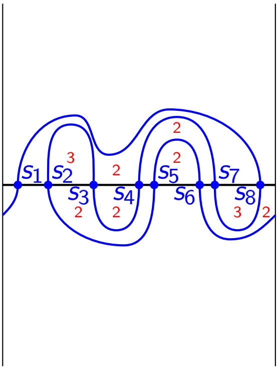
  $\mathcal{A}(q)$ is a lower end of a bar $\Leftrightarrow \partial(q)$ is a boundary in $\mathrm{CF}^{<\mathcal{A}(q)}(L, L')$

- If $[\mathcal{A}(p), \mathcal{A}(q))$ is a bar, then p occurs as a summand in $\partial(q')$ for some q' with $\mathcal{A}(q') \leq \mathcal{A}(q)$.
- If $\partial(q)$ contains p as a summand, then there is a bar contained in the interval $[\mathcal{A}(p), \mathcal{A}(q))$.

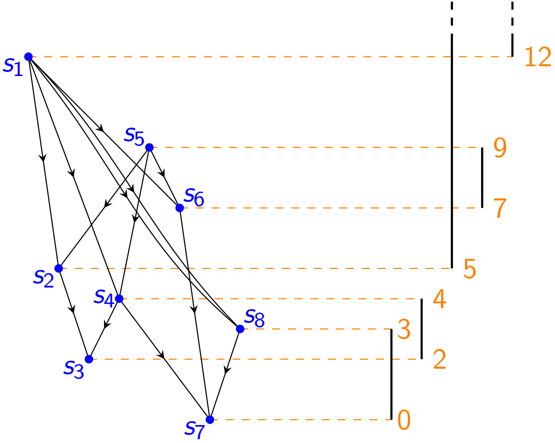
Example: $\mathcal{B}(L, L')$



(L, L') $CF(L, L')$ $\mathcal{B}(L, L')$



(L, L')



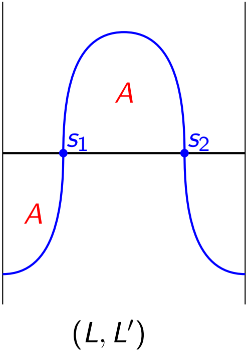
$CF(L, L')$ $\mathcal{B}(L, L')$

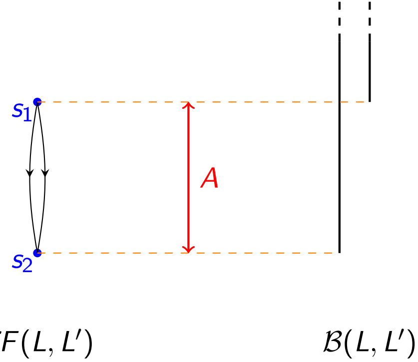
Base case of induction

Given: L, L' intersecting transversely in $2n$ points ($n \geq 1$)

Strategy: Induction on n

For $n = 1$: $d_H(L, L') = A = \gamma(L, L')$





Patricia Dietzsch
Doctoral student at ETH
patricia.dietzsch@math.ethz.ch

ETH Zürich
D-MATH
HG FO 28.7
Rämistrasse 101
8092 Zurich, SWITZERLAND
https://people.math.ethz.ch/~pdietzsc/