Isolated hypersurface singularities, spectral invariants, and quantum cohomology

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Symplectic Zoominar
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Theme: Study algebraic & symplectic geometry (AG & SG) of singularities via spectral invariants (some symplectic invariant coming from Floer theory).
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spectral invariants.
Singularities

Algebraic geometers study algebraic varieties. Varieties are not necessarily smooth, i.e. they can have singularities. Isolated hypersurface singularities form a fundamental and important class of singularities. Isolated hypersurface singularities were classified by Arnold (up to "modality" two):

- modality zero: simple singularities (ADE)
- modality one: parabolic ($E_6 \hookrightarrow E_7 \hookrightarrow E_8$), hyperbolic ($T_{p,q,r}$), the 14 exceptional singularities.

Higher modality ones.

In this talk, singular varieties all assumed to have at most isolated hypersurface singularities.
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Degeneration

Important question in AG: If a smooth (Fano) variety \( X \) degenerates to a singular variety \( X_0 \), what types of singularities can \( X_0 \) have?

We say "\( X \) degenerates to a singular variety \( X_0 \)" if the following happens:

**Definition**

Let \( X \) be a smooth (Fano) variety. A degeneration of \( X \) is a flat family \( \phi: X \to C \) such that:

- The only singular fiber is \( X_0 := \phi^{-1}(0) \).
- The variety \( X \) is smooth away from the singular locus of \( X_0 \).
- Some regular fiber is \( X \).

In AG, understanding the types of singularities that can occur on a variety \( X \) is very important, c.f. minimal model program, enumerative geometry, etc.
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Degeneration in SG

Degeneration is a notion in AG (up to here, there is no SG). To make connection to SG, we need to be in a “favourable (algebro-geometric) situation” (e.g. if there exists a $\pi$-relative ample line bundle $L \to X$).

Then one has a family of projective embedding $f_t: X_t, \to \mathbb{C}P^N$ and we can start seeing varieties $X_t$ as symplectic manifolds ($X_t \hookrightarrow !t := f^*t!FS$).

Moreover, you can define symplectic parallel transport in the total space $X$ can define vanishing cycles. Arnold, Donaldson noticed that the vanishing cycles of the singularities in $X_0$ can give Lagrangian spheres in the regular fibers ($X_t \hookrightarrow !t, t \neq 0$ [provide within a “favourable situation”]).

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- Arnold, Donaldson noticed that the vanishing cycles of the singularities in $X_0$ can give Lagrangian spheres in the regular fibers
  $(X_t, \omega_t)$, $t \neq 0$ (provided that we are in a “favorable situation”).
For example, the vanishing cycles of simple singularities, i.e. ADE, give collections of Lagrangian spheres as the ADE Dykin diagrams:

\[ \begin{align*}
A_n & \quad \bullet \ldots \bullet \\
D_n & \quad \bullet \ldots \bullet \\
E_6 & \quad \bullet \\
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**Figure:** Dynkin diagrams of type \( A_n, D_n, E_6, E_7, E_8 \).
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**Figure**: Dynkin diagrams of type \( A_n, D_n, E_6, E_7, E_8 \).

- 2-dim. has been studied a lot, but Arnold emphasized the importance/interest of studying high dimensional cases of singularities.
Quantum cohomology ring

\( \mathbb{QH}(X) = \mathbb{H}^*(X; \mathbb{C}) \oplus \mathbb{C}^\infty \) where \( \mathbb{C}^\infty \) is the universal Novikov field (à la FOOO).

An interesting case: when \( \mathbb{QH} \) is semi-simple.
Quantum cohomology ring

- QH (quantum cohomology ring) is another topic studied both in AG and SG. (c.f. idea comes from Vafa, Witten, AG-formulation by Kontsevich–Manin (94), SG-formulation by Ruan–Tian (95)).
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\[ QH(X, \omega) := H^*(X; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda \]

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- An interesting case: when QH is semi-simple.
Recall that \( \text{QH} \) is semi-simple when it splits into a direct sum of fields:

\[
\text{QH} \cong \bigoplus_{j} \text{Q}_j
\]

where each \( \text{Q}_j \) is a field (over \( \mathbb{C} \)).

Once again, semi-simplicity is of interest for AG and SG communities, e.g. Bayer–Manin, Dubrovin for AG, Entov–Polterovich for SG.

Monotone examples:
- \( \mathbb{C}P^n \), the quadric hypersurface \( Q^n \), del Pezzo surfaces \( D_k := \mathbb{C}P^2 \# k \cdot (\mathbb{C}P^2) \) (degree 9 \( k \)), with \( 0 < k < 4 \),
- complex Grassmannians \( \text{Gr}_{\mathbb{C}}(k \hookrightarrow n) \), their products.

"Generic" examples:
- Toric Fano varieties (FOOO, Ostrover–Tyomkin, Usher),
- Many (36/59) of the Fano 3-folds (Ciolli),
- their one-point blow ups (Usher).
Recall that $QH$ is semi-simple when it splits into a direct sum of fields:

$$QH(X, \omega) = \bigoplus_{1 \leq j \leq k} Q_j$$

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Theorem A: Hypersurface singularities and QH (AG)

No relation between isolated hypersurface singularities and QH seems to be known.

Theorem A: AG formulation (K.)

Let $X$ be a complex $n$-dimensional smooth Fano variety with even $n$.

Assume either one of the following two:

1. $\text{QH}(X \hookrightarrow !)$ is semi-simple where $!$ is the natural symplectic form coming from the projective embedding of $X$.
2. $n > 2$ and the quantum cohomology ring is generically semi-simple.

If $X$ degenerates to a Fano variety with an isolated hypersurface singularity, then the singularity has to be of type A.

In fact, to prove Theorem A (AG), we reduce it to its "symplectic-counterpart" Theorem A' (SG), but this "translation" is NOT immediate.
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In fact, to prove Theorem A (AG), we reduce it to its “symplectic-counterpart” Theorem A’ (SG), but this “translation” NOT immediate.
Let \((X, \omega)\) be a real \(2n\) dimensional closed symplectic manifold with even \(n\). If \(QH(X, \omega)\) is semi-simple, then \((X, \omega)\) cannot contain a configuration of Lagrangian spheres coming from an isolated hypersurface singularity that is not of type A.

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- **(CY-case)** It is well-known that $D, E, 14$ exceptional singularities can appear in the degeneration of the K3 surface ($QH(K3)$ not semi-simple).

Unlike related AG-results, Theorem A has the advantage of not having any low-dimensional constraints, as our argument is SG-based (matches Arnold's perspective on higher dimensions).
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Relation to other works: *Compactification of Milnor fibers. (SG)*

The following is an immediate consequence of Theorem A':

**Corollary**

The Milnor fiber of an isolated hypersurface singularity that is not of A-type cannot be compactified to a symplectic manifold with semi-simple quantum cohomology ring.

The following two are compatible with this:

- Keating compactifies Milnor fibers of $\mathfrak{e}_6 \hookrightarrow \mathfrak{e}_7 \hookrightarrow \mathfrak{e}_8$ to $\mathfrak{d}_6 \hookrightarrow \mathfrak{d}_7 \hookrightarrow \mathfrak{d}_8$, respectively (*QH*($\mathfrak{d}_k$) not semi-simple).

- Dolgachev, Nikulin, Pinkham compactifies Milnor fibers of the 14 exceptional singularities to K3 surface (*QH*($\mathfrak{k}_3$) not semi-simple).
The following is an immediate consequence of Theorem A’:

\[ e^{E_6} \hookrightarrow e^{E_7} \hookrightarrow e^{E_8} \to D_6 \hookrightarrow D_7 \hookrightarrow D_8, \]

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- Dolgachev, Nikulin, Pinkham compactifies Milnor fibers of the 14 exceptional singularities to K3 surface ($QH(K3)$ not semi-simple).
Theorem B: A-type configuration and Hofer geometry

Theorem A' excludes D, E, etc., type configurations in $(X \hookrightarrow \mathfrak{g})$ with semi-simple $QH$, but it is still possible to have A type configurations. In fact, this can happen, e.g. del Pezzo surfaces $D_k$ with $0 \leq k \leq 4$, etc. If it happens, we have the following implication on the Hofer geometry:

Theorem B (K.)

Let $(X \hookrightarrow \mathfrak{g})$ be a real $2n$-dimensional closed symplectic manifold with even $n$. Assume $QH(X \hookrightarrow \mathfrak{g})$ is semi-simple. If $(X \hookrightarrow \mathfrak{g})$ contains an $A_m$-configuration of Lagrangian spheres, then there are $m$ linearly independent Entov–Polterovich quasimorphisms on $\text{Ham}(X \hookrightarrow \mathfrak{g})$. 

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If it happens, we have the following implication on the Hofer geometry (will not review Hofer geometry, but will be out of this topic shortly):

**Theorem B (K.)**

Let \((X \hookrightarrow \mathcal{M})\) be a real \(2n\)-dimensional closed symplectic manifold with even \(n\).

Assume \(QH(X \hookrightarrow \mathcal{M})\) is semi-simple. If \((X \hookrightarrow \mathcal{M})\) contains an \(A_m\)-configuration of Lagrangian spheres, then there are \(m\) linearly independent Entov–Polterovich quasimorphisms on \(\text{Ham}(X \hookrightarrow \mathcal{M})\).
Theorem B: A-type configuration and Hofer geometry

- Theorem A’ excludes D, E, etc., type configurations in \((X, \omega)\) with semi-simple QH, but it is still possible to have A type configurations.
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Let $(X, \omega)$ be a real $2n$ dimensional closed symplectic manifold with even $n$. Assume $QH(X, \omega)$ is semi-simple. If $(X, \omega)$ contains an $A_m$-configuration of Lagrangian spheres, then there are $m - 1$ linearly independent Entov–Polterovich quasimorphisms on $\widehat{\text{Ham}}(X, \omega)$. 

Yusuke Kawamoto (ETH Zürich)
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**Corollary (Kapovich–Polterovich question) (K.)**

There are four linearly independent Entov–Polterovich quasimorphisms on $\text{Ham}(D^4)$. Thus, $\text{Ham}(D^4)$ admits a quasi-isometric embedding of $\mathbb{R}^4$. In particular, the group $\text{Ham}(D^4)$ is not quasi-isometric to the real line $\mathbb{R}$ with respect to the Hofer metric.
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Key ingredients of the proof

Keys of the proofs of Theorems A’, B, (C):

1. Entov–Polterovich’s asymptotic spectral invariant theory (quasimorphisms, (super)heaviness).
2. Biran–Membrez’s Lagrangian cubic equation.
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- We give a quick overview of the two.
Quick review: spectral invariants

Entov–Polterovich's theory of asymptotic spectral invariants (quasimorphisms, (super)heaviness) is based on Viterbo's spectral invariant theory. Spectral invariants are powerful tools to study the Hofer geometry/Hamiltonian dynamics (introduced by Viterbo, developed by Schwarz, Oh, Leclercq etc.).

Given a Hamiltonian $H$, one can define a filtered Floer complex $\text{CF} \ni (H)$ $\hookrightarrow \pi_2 \mathbb{R}$ (=generators are periodic orbits with action $\notin \pi_2 \mathbb{R}$).

This gives you a filtered Floer homology $\text{HF} \ni (H)$. The inclusion induces the map $i \ni (H) : \text{HF} \ni (H) \rightarrow \text{HF} (H)$.

We also have the PSS map $\text{PSS} H : QH (X \hookrightarrow !) \rightarrow \text{HF} (H)$.

We define a spectral invariant for a pair of a Hamiltonian $H$ and a class $a \in QH (X \hookrightarrow !)$ as follows:

$c(H \hookrightarrow a) = \inf \{ \pi_2 \mathbb{R} : \text{PSS} H (a) \subseteq \text{Im} (i \pi_2 \mathbb{R}) \}$. 

Yusuke Kawamoto (ETH Zürich)
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[Equation and discussion]

Yusuke Kawamoto (ETH Zürich)
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Entov–Polterovich introduced a notion of symplectic rigidity called the (super)heaviness by using Hamiltonian spectral invariants.

- **Superheaviness**
  - Suppose $QH(X \hookrightarrow !)$ has a field summand: $QH(X \hookrightarrow !) = \mathbb{Q}[A]$ with $\mathbb{Q}$: field. Decompose the unit $1_X$ with respect to this split: $1_X = e + e^0$.
  - Then, the asymptotic spectral invariant of $e$ is the following: $\downarrow e: C_1(X) \to \mathbb{R} \downarrow e(\mathcal{H}) := \lim_{k \to +1} c(k \cdot H \mapsto e)k$.

- As subset $S \subseteq X$ is superheavy wrt. the idempotent $e$ iff for any $H$, we have $\inf_{x \in S} H(x) \leq \downarrow e(\mathcal{H}) \leq \sup_{x \in S} H(x)$.
One can do the same for Lagrangian Floer homology of a Lagrangian \( L \) and define \( \ell_L(H) \) (spectral invariant for \( 1_L \in HF(L) \)).

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**Definition: Superheaviness**

Suppose \( QH(X, \omega) \) has a field summand: \( QH(X, \omega) = Q \oplus A \) with \( Q \): field. Decompose the unit \( 1_X \) with respect to this split: \( 1_X = e + e' \).
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$$\zeta_e : C^\infty(X) \to \mathbb{R}$$

$$\zeta_e(H) := \lim_{k \to +\infty} \frac{c(k \cdot H, e)}{k}.$$
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A subset \( S \subset X \) is superheavy wrt. the idempotent \( e \) iff for any \( H \), we have

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\inf_{x \in S} H(x) \leq \zeta_e(H) \leq \sup_{x \in S} H(x).
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(side remark) We can also define the asymptotic Lagrangian spectral invariants:

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The following is a useful property of superheaviness:

**Lemma: no disjoint e-superheavy sets**

Two disjoint subsets, say \(A\) and \(B\), can never be superheavy with respect to the same idempotent \(e\).
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- This is easy: Assume disjoint subsets $A$ and $B$ are both superheavy with respect to the same idempotent $e$.
- Just take a $H$ such that $H|_A \equiv 0$ and $H|_B \equiv 1$. 
(side remark) We can also define the asymptotic Lagrangian spectral invariants:

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Just take a \(H\) such that \(H|_A \equiv 0\) and \(H|_B \equiv 1\).

Then, we have

\[
1 = \inf_{x \in B} H(x) \leq \zeta_e(H) \leq \sup_{x \in A} H(x) = 0,
\]

which is a contradiction. Proof done.
Let $L$ be a Lagrangian sphere in a real $2n$-dimensional closed symplectic manifold $(X \hookrightarrow !)$. See the (co)homology class $[L]$ as a class in $\mathbb{Q}H(X \hookrightarrow !)$. It satisfies the following equation:

$$[L]^3 = 4[L]$$

for some $L^2 \in \mathbb{Q}H(X \hookrightarrow !)$. If $L = 0$, then $[L]^2 \in \mathbb{Q}H(X \hookrightarrow !)$ is nilpotent. If $L \neq 0$, then the cubic equation implies that the following two are idempotents of $\mathbb{Q}H(X \hookrightarrow !)$:

$$e^L = \pm 1 - \frac{1}{4}L + \frac{1}{8}L^2.$$ 

Moreover, $e^L$ are units of field factors of $\mathbb{Q}H(X \hookrightarrow !)$, i.e. $e^L \cdot \mathbb{Q}H(X \hookrightarrow !) = \mathbb{Q}H(X \hookrightarrow !)$. 

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Yusuke Kawamoto (ETH Zürich)

Symplectic Zoominar 17 February, 2023
Let $L$ be a Lagrangian sphere in a real $2n$ dimensional closed symplectic manifold $(X, \omega)$ with even $n$. See the (co)homology class $[L]$ as a class in $QH(X, \omega)$.

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Moreover, $e_L \pm$ are units of field factors of $QH(X, \omega)$, i.e.

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Biran–Membrez’s Lagrangian cubic equation

Let $L$ be a Lagrangian sphere in a real $2n$ dimensional closed symplectic manifold $(X, \omega)$ with even $n$. See the (co)homology class $[L]$ as a class in $QH(X, \omega)$. It satisfies the following equation:

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for some $\beta_L \in \Lambda$. 
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- If $\beta_L = 0$, then $[L] \in QH(X, \omega)$ is nilpotent.
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Moreover, $e^L_\pm$ are units of field factors of $QH(X, \omega)$, i.e.

$$e^L_\pm \cdot QH(X, \omega) = \Lambda.$$
• Thus, if $\beta_L \neq 0$, we get $\zeta_{e_L}$.
Thus, if $\beta_L \neq 0$, we get $\zeta_{e^\pm}$.

If $QH(X, \omega)$ is semi-simple, there are no nilpotents, so $\beta_L \neq 0$. 
Thus, if $\beta_L \neq 0$, we get $\zeta_{e_L}^\pm$.

If $QH(X, \omega)$ is semi-simple, there are no nilpotents, so $\beta_L \neq 0$.

Thus, when $QH(X, \omega)$ is semi-simple, we always have $\zeta_{e_L}^\pm$. (From now on, we always assume $QH$ is semi-simple.)
Two lemmas

Lemma 1: idempotent sharing property
If two Lagrangian spheres $L$ and $L_0$ are intersecting, then we have

$$L = L_0 \quad (6 = 0)$$

don't forget it should be shared between $L$ and $L_0$:

$$\varepsilon_L + \varepsilon_{L_0} = \varepsilon_{L_0}.$$ 

Lemma 2: $L$ is $\varepsilon_L \pm$-superheavy
We have the following relation between Hamiltonian and Lagrangian spectral invariants of a Lagrangian sphere $L$ with $L_6 = 0$:

$$L(H) = \max \left\{ \varepsilon_{L \pm}(H) \right\}.$$

In particular, $L$ is $\varepsilon_L \pm$-superheavy, i.e. superheavy with respect to both.
Lemma 1: idempotent sharing property

If two Lagrangian spheres $L$ and $L'$ are intersecting, then we have $\beta_L = \beta_{L'} (\neq 0)$, and one of the two corresponding idempotents should be shared between $L$ and $L'$:

$$e^L_+ = e^{L'}_-. $$
Two lemmas

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We have the following relation between Hamiltonian and Lagrangian spectral invariants of a Lagrangian sphere $L$ with $\beta_L \neq 0$:

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In particular, $L$ is $e_L^\pm$-superheavy, i.e. superheavy with respect to both $e_L^\pm$. 
Proof of Theorem A’

We start with the simple singularity (ADE) case. Suppose there is a D or E configuration of Lagrangian spheres in \((X \hookrightarrow !)\).

In either case, there is a Lagrangian sphere \(S\) that intersects three other Lagrangian spheres \(S_1 \hookrightarrow S_2 \hookrightarrow S_3\).
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Figure: Dynkin diagrams of type $A_n, D_n, E_6, E_7, E_8$. 
Proof of Theorem A’

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**Figure:** Dynkin diagrams of type \(A_n, D_n, E_6, E_7, E_8\).

- In either case, there is a Lagrangian sphere \(S\) that intersects three other Lagrangian spheres \(S_1, S_2, S_3\).
2 ≠ 3.
By the “idempotent sharing lemma” (Lemma 1), we have that the two idempotents of $S$, i.e. $e^S_\pm$, has to be shared with $S_1$, $S_2$, and $S_3$.

By similar argument, no higher modality configurations can appear! Proof done.
By the “idempotent sharing lemma” (Lemma 1), we have that the two idempotents of $S$, i.e. $e^S_\pm$, has to be shared with $S_1$, $S_2$, and $S_3$. (what I mean by this, is something like $e^S_\pm = e^{S_1}_\pm$, $e^S_\pm = e^{S_2}_\pm$, $e^S_\pm = e^{S_3}_\pm$ is happening.)
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- As $2$ (number of idempotents produced by $S$) $\neq 3$ (number of spheres intersecting $S$), one of the $e^S_\pm$ has to be shared by two of $S_1$, $S_2$, $S_3$. 
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But the spheres $S_1$, $S_2$, $S_3$ are all disjoint! How come two of them can share an idempotent ($e^S$ or $e_-^S$) for which they are superheavy (Recall the “non disjoint superheavy lemma”)? This is a contradiction!
By the “idempotent sharing lemma” (Lemma 1), we have that the two idempotents of $S$, i.e. $e^S_\pm$, has to be shared with $S_1, S_2,$ and $S_3$.

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Conclusion: there cannot be any DE configurations.
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*higher modality case*: the corresponding Dynkin diagrams for these singularities are known thanks to Gabrielov, Keating.
By the “idempotent sharing lemma” (Lemma 1), we have that the two idempotents of $S$, i.e. $e^S_{\pm}$, has to be shared with $S_1$, $S_2$, and $S_3$. (what I mean by this, is something like $e^S_- = e^{S_1}_+$, $e^S_+ = e^{S_2}_-$, $e^S_+ = e^{S_3}_-$ is happening.)

As 2 (number of idempotents produced by $S$) $\neq 3$ (number of spheres intersecting $S$), one of the $e^S_{\pm}$ has to be shared by two of $S_1$, $S_2$, $S_3$.

But the spheres $S_1$, $S_2$, $S_3$ are all disjoint! How come two of them can share an idempotent ($e^S$ or $e^-_S$) for which they are superheavy (Recall the “non disjoint superheavy lemma”)? This is a contradiction!

Conclusion: there cannot be any DE configurations.

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By similar argument, no higher modality configurations can appear!
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Proof done.
Theorem C: Dehn twist and spectral invariants

Recall that configurations were the starting point of the study of Dehn twist by Seidel (Arnold).

Question: What effect does the Dehn twist have on spectral invariants?

Theorem C (K.)

Let \((X \hookrightarrow !)\) be a real 2-dimensional closed symplectic manifold with even \(n\). Assume \(\text{QH}(X \hookrightarrow !)\) is semi-simple. If \((X \hookrightarrow !)\) contains an \(A_2\) configuration, i.e. two Lagrangian spheres \(L \hookrightarrow L_0\) with \(|L \setminus L_0| = 1\), then we have

\[
\langle \tau L \rangle (H) \leq \max \{ \langle \tau L_0 \rangle (H) \}
\]

for any Hamiltonian \(H\), where \(\tau L\) is the Dehn twist about \(L\).
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Yusuke Kawamoto (ETH Zürich)
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**Theorem C (K.)**
Let $(X, \omega)$ be a real $2n$ dimensional closed symplectic manifold with even $n$. Assume $QH(X, \omega)$ is semi-simple. If $(X, \omega)$ contains an $A_2$ configuration, i.e. two Lagrangian spheres $L, L'$ with $|L \cap L'| = 1$, then we have

$$\bar{\ell}_{\tau_L}(L')(H) \leq \max\{\bar{\ell}_L(H), \bar{\ell}_{L'}(H)\}$$

for any Hamiltonian $H$, where $\tau_L$ is the Dehn twist about $L$. 
Key of the proof of Theorem C (Dehn twists and spectral invariants)

Recall that we had

\[ e^L = \pm \frac{1}{4} p^L L^2. \]

(Dehn twist swaps the idempotent) By using the Picard–Lefschetz formula, we can express \( \tau^L \) and by plugging this into the formula of \( e^\tau^L (L^0) \leftarrow \), we get

\[ e^\tau^L (L^0) \leftarrow = e^L \leftarrow e^L^0 + \ldots. \]

Combine it with the previous lemma

\[ L(H) = \max \{ e^L \pm (H) \}. \]
Key of the proof of Theorem C (Dehn twists and spectral invariants)

- Recall that we had

\[ e^L_{\pm} = \pm \frac{1}{4\sqrt{\beta_L}}[L] + \frac{1}{8\beta_L}[L]^2. \]
Key of the proof of Theorem C (Dehn twists and spectral invariants)

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Key of the proof of Theorem C (Dehn twists and spectral invariants)

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  \[ e^\tau_{L'}(L') = e^L_{-}, e^L_{+}. \]

- Combine it with the previous lemma
  \[ \ell_L(H) = \max \zeta e^L_{\pm}(H). \]
Summary

1. You can prove AG-results by using spectral invariants (Theorems A&A').
2. AG (namely singularities) can tell something about Hofer geometry (Theorem B).
3. Dehn twist reduces the spectral invariant (Theorem C).
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2. AG (namely singularities) can tell something about Hofer geometry (Theorem B).

3. Dehn twist reduces the spectral invariant (Theorem C).
Thank you very much for your attention!