Isolated hypersurface singularities, spectral invariants, and quantum cohomology

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Pre-Intro

- Theme: Study algebraic & symplectic geometry (AG & SG) of singularities via spectral invariants (some symplectic invariant coming from Floer theory).

Introduction

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- In this talk, singular varieties all assumed to have at most isolated hypersurface singularities.

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Let X be a smooth (Fano) variety. A degeneration of X is a flat family $\pi : \mathcal{X} \to \mathbb{C}$ such that

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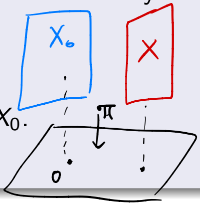

- In AG, understanding the types of singularities that can occur on a variety X is very important, c.f. minimal model program, enumerative geometry, etc.

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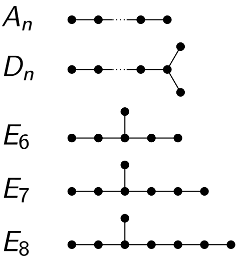
- Moreover, you can define symplectic parallel transport in the total space $\mathcal{X}$ can define vanishing cycles.
- Arnold, Donaldson noticed that the vanishing cycles of the singularities in X_0 can give Lagrangian spheres in the regular fibers (X_t, ω_t), $t \neq 0$ (provided that we are in a "favorable situation").

- For example, the vanishing cycles of simple singularities, i.e. ADE, give collections of Lagrangian spheres as the ADE Dykin diagrams:



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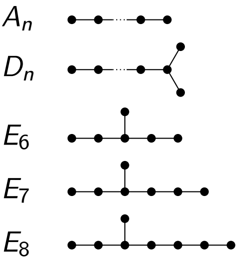

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- 2-dim. has been studied a lot, but Arnold emphasized the importance/interest of studying high dimensional cases of singularities.

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- An interesting case: when QH is semi-simple.

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- Monotone examples:
  - $\mathbb{C}P^n$, the quadric hypersurface Q^n,
  - del Pezzo surfaces $\mathbb{D}_k := \mathbb{C}P^2 \# k \cdot (\overline{\mathbb{C}P^2})$, (degree $9-k$), with $0 \leqslant k \leqslant 4$,
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- "Generic" examples:
  - Toric Fano varieties (FOOO, Ostrover–Tyomkin, Usher),
  - Many (36/59) of the Fano 3-folds (Ciolli),
  - their one-point blow ups (Usher).

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Let X be a complex n dimensional smooth Fano variety with even n. Assume either one of the following two:

- $QH(X, \omega)$ is semi-simple where ω is the natural symplectic form coming from the projective embedding of X.
- $n > 2$ and the quantum cohomology ring is generically semi-simple.

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- In fact, to prove Theorem A (AG), we reduce it to its "symplectic-counterpart" Theorem A' (SG), but this "translation" NOT immediate.

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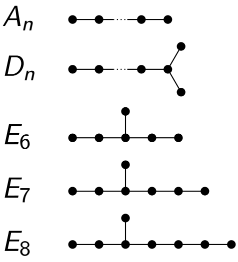
Let (X, ω) be a real $2n$ dimensional closed symplectic manifold with even n. If $QH(X, \omega)$ is semi-simple, then (X, ω) cannot contain a configuration of Lagrangian spheres coming from an isolated hypersurface singularity that is not of type A.



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Unlike related AG-results, Theorem A has has the advantage of not having any low-dimensional constraints, as our argument is SG-based (matches Arnold's perspective on higher dimensions).

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- Dolgachev, Nikulin, Pinkham compactifies Milnor fibers of the 14 exceptional singularities to K3 surface ($QH(K3)$ not semi-simple).

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Corollary (Kapovich–Polterovich question) (K.)

There are four linearly independent Entov–Polterovich quasimorphisms on $\mathrm{Ham}(\mathbb{D}_4)$. Thus, $\mathrm{Ham}(\mathbb{D}_4)$ admits a quasi-isometric embedding of $\mathbb{R}^4$. In particular, the group $\mathrm{Ham}(\mathbb{D}_4)$ is not quasi-isometric to the real line $\mathbb{R}$ with respect to the Hofer metric.

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- We give a quick overview of the two.

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- This gives you a filtered Floer homology $HF^{\tau}(H)$. The inclusion induces the map $i^{\tau} : HF^{\tau}(H) \to HF(H)$.
- We also have the PSS map $PSS_H : QH(X, \omega) \to HF(H)$.

Quick review: spectral invariants

- Entov–Polterovich's theory of asymptotic spectral invariants (quasimorphisms, (super)heaviness) is based on Viterbo's spectral invariant theory.
- Spectral invariants are powerful tool to study the Hofer geometry/Hamiltonian dynamics (introduced by Viterbo, developed by Schwarz, Oh, Leclercq etc.).
- Given a Hamiltonian H, one can define a filtered Floer chain complex $CF^{\tau}(H)$, $\tau \in \mathbb{R}$ (=generators are periodic orbits with action $\leqslant \tau$).
- This gives you a filtered Floer homology $HF^{\tau}(H)$. The inclusion induces the map $i^{\tau} : HF^{\tau}(H) \to HF(H)$.
- We also have the PSS map $PSS_H : QH(X, \omega) \to HF(H)$.
- We define a spectral invariant for a pair of a Hamiltonian H and a class $a \in QH(X, \omega)$ as follows:

$$c(H, a) := \inf\{\tau \in \mathbb{R} : PSS_H(a) \in Im(i^{\tau})\}.$$

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Suppose $QH(X,\omega)$ has a field summand: $QH(X,\omega) = Q \oplus A$ with Q: field. Decompose the unit 1_X with respect to this split: $1_X = e + e'$.

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$$\zeta_e : C^\infty(X) \to \mathbb{R}$$

$$\zeta_e(H) := \lim_{k \to +\infty} \frac{c(k \cdot H, e)}{k}.$$

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A subset $S \subset X$ is superheavy wrt. the idempotent e iff for any H, we have

$$\inf_{x \in S} H(x) \leqslant \zeta_e(H) \leqslant \sup_{x \in S} H(x).$$

- (side remark) We can also define the asymptotic Lagrangian spectral invariants:

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- Then, we have

$$1 = \inf_{x \in B} H(x) \leqslant \zeta_e(H) \leqslant \sup_{x \in A} H(x) = 0,$$

which is a contradiction. Proof done.

Biran–Membrez's Lagrangian cubic equation

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- Moreover, $e_{\pm}^L$ are units of field factors of $QH(X, \omega)$, i.e. $e_{\pm}^L \cdot QH(X, \omega) = \Lambda$.

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- If $QH(X,\omega)$ is semi-simple, there are no nilpotents, so $\beta_L \neq 0$.
- Thus, when $QH(X,\omega)$ is semi-simple, we always have $\zeta_{e^L_\pm}$. (From now on, we always assume QH is semi-simple.)

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Two lemmas

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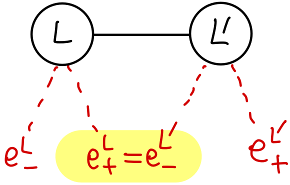
Lemma 1: idempotent sharing property

If two Lagrangian spheres L and L' are intersecting, then we have $\beta_L = \beta_{L'} (\neq 0)$, and one of the two corresponding idempotents should be shared between L and L':

$$e_+^L = e_-^{L'}.$$



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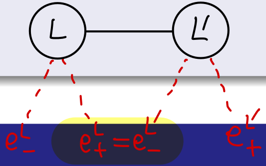

Lemma 2: L is $e_\pm^L$-superheavy

We have the following relation between Hamiltonian and Lagrangian spectral invariants of a Lagrangian sphere L with $\beta_L \neq 0$:

$$\bar{\ell}_L(H) = \max \zeta_{e_\pm^L}(H).$$

In particular, L is $e_\pm^L$-superheavy, i.e. superheavy with respect to both $e_\pm^L$.

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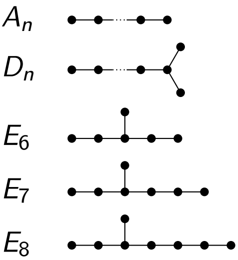

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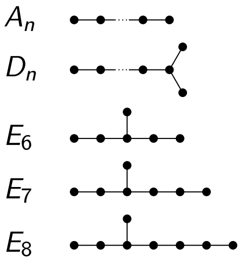

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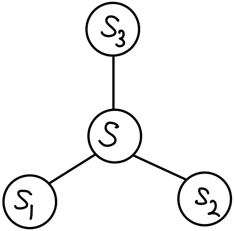

- In either case, there is a Lagrangian sphere S that intersects three other Lagrangian spheres S_1, S_2, S_3.

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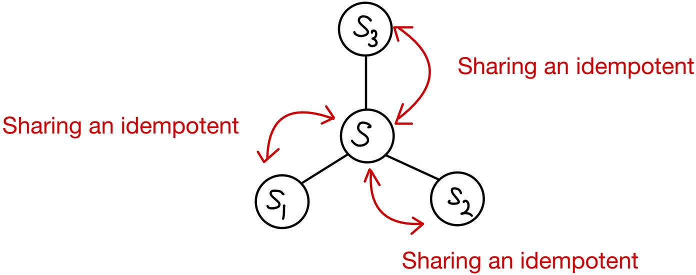

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Theorem C (K.)

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$$\overline{\ell}_{\tau_L(L')}(H) \leqslant \max\{\overline{\ell}_L(H), \overline{\ell}_{L'}(H)\}$$

for any Hamiltonian H, where τ_L is the Dehn twist about L.

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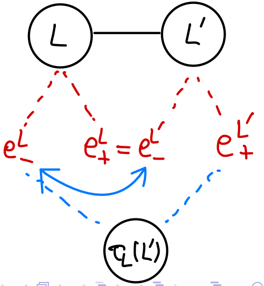

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- Combine it with the previous lemma

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2. AG (namely singularities) can tell something about Hofer geometry (Theorem B).
3. Dehn twist reduces the spectral invariant (Theorem C).

Thank you very much for your attention!