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Floer theory &
Framed cobordism between exact Lagrangians

Based on joint work in progress with Ivan Smith

1. Background + questions

2. Framed bordism

3. Floer theory.
1. Background + Qs.

- \( (X^{2n}, \omega) \) symplectic mfd.

- Today: \( X \) always Liouville
  - non-compact
  - \( \omega = \lambda \)
  - nice @ \( \infty \)

Examples

- \( X = T^*Q \)
- \( X = T^*Q, \# T^*Q, \# T^*Q^3 \)
- smooth cx affine variety.
Question: What are the diffeo classes of closed exact lags $L \subseteq \mathcal{X}$

- $\dim L = \lambda = \frac{1}{2} \dim \mathcal{X}$

- $\lambda \in \mathcal{M}'(\mathcal{L})$ exact.

Nearby Lag conjecture

$\implies$ If $X = T^*Q$
then any such $L \subseteq X$ is diffeo to $Q$. 
Theorem (Abouzaid, Grghi, Guillermay)

If $x = T^*Q$

any closed exact $L \leq x$

is $\sim Q$.

$\Rightarrow$ Some $H_+, T_x$

How different can $\sim$ manifolds be?

Example (Serre vanre - Milnor)

There are exactly $2^8$ different oriented classes of manifolds which $\sim S^7$.

$1 \sim 2 \sim \cdots \sim S^8$.

All of these are homeomorphic.
2. Framed bundles

**Def.** A **stable framing** on a manifold $M^n$ is an element of $\text{v.b.s.}$ $TM \oplus \mathbb{R}^k \cong \mathbb{R}^{n+k}$.

- A **stably framed manifold** is a pair $(M, \xi)$, where $\xi$ is a stable framing on $M$.

- Given $M$, there might be no $\xi$ or $\xi$ but not unique.
Examples

- $S^n \to \mathbb{R}^{n+1}$

$$TS^n \oplus \mathbb{R}^1 \cong \mathbb{R}^{n+1}$$

stable framing.

- $S^1, S^3$ with the Lie gp framing

$$TS^1 \cong \mathbb{R}^1$$

$$TS^3 \cong \mathbb{R}^3$$

NOT the same as

- If $M$ has a stable framing, all $p_i(M) = 0$,
  $\implies \mathbb{RP}^2, \mathbb{CP}^2$ don't have.
**Def.** A **framed bordism** \((W, \sigma)\) from \((M^3, \xi)\) to \((M', \xi')\) is cobordism \(W\) from \(M\) to \(M'\) and a stable framing \(\sigma\) s.t. 

\[
\sigma|_M = \xi \\
\sigma|_{M'} = -\xi'
\]

• Framed bordism is an equivalence relation.
Def. The framed bordism groups
\[ \Omega^f_n = \{ \text{stably framed } n \text{-manifolds} \} \]

\[ \Omega^f_n \quad \text{framed bordism} \]

- Abelian gp under \( \cup \)
- Unit is \( \emptyset \)
- \( \Omega^f_n \) is a graded ring
  product is product of mflds
<table>
<thead>
<tr>
<th>n</th>
<th>$\mathbb{Z}^n$</th>
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<tbody>
<tr>
<td>0</td>
<td>$\mathbb{Z}$</td>
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<tr>
<td>1</td>
<td>$\mathbb{Z}/2$</td>
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<tr>
<td>2</td>
<td>$\mathbb{Z}/2$</td>
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<tr>
<td>$\vdots$</td>
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</tr>
<tr>
<td>7</td>
<td>$\mathbb{Z}/2^4$</td>
</tr>
<tr>
<td>8</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
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- Any exotic sphere admit a stable framing.
- Let $E = \text{the unique exotic } S^8$.
- $[E] \neq [S^8]$ in $\mathbb{Z}_8^{fr}$ for any stable framings.

Generated by $(S^1, \text{Lie})$.
Let \((X^{2n}, \omega = d\alpha)\) be Liouville.

- Let \(L, k \subseteq X\) be 2 closed exact Lag.s
  
  Then (A. Smith, in progress).

Assume

- \(L, k\) are homology spheres.

- \(L, k\) have the same Floer theory over \(\mathbb{R}\).
  
  \((\Rightarrow \cong \text{ in Fukaya cat})\)

\[\begin{aligned}
\text{TX stably trivial} \\
\text{TX } \bigoplus \mathbb{C}^k = \mathbb{C}^{n+k} \quad \text{some } k \hspace{1cm} \text{Pick a } a \in \\
\text{TL } \bigoplus \mathbb{R}^k = \mathbb{R}^{n+k} \quad \text{stably framed too}
\end{aligned}\]
Then $[L] - [k] \in \mathcal{M}^f_\eta$

is 2-torsion.

$[L] - [k] \in \mathcal{M}^f_\eta$. \mathcal{M}^f_\eta \subseteq \mathcal{M}^f_\eta

\Rightarrow \mathbb{Z}/2

Cor 1 (Abouzaid-Alvarez-Govens-Courte-Kragh)

If $L \subseteq \ast^\# S^7$ closed exact log

Then $[L] \in \mathcal{M}^f_\eta \cdot \mathcal{M}^f_\eta$

$\Rightarrow$ If $\eta = 8$, $L$ diffeo to $S^8$. 
Cor 2  Let $X = T_S^n \# T_S^n$, and $L \subset X$ a closed exact Lag.

Assume:  
  * Maslov class of $L = 0$.  
  * $\nu \equiv 0, 9, 6, 7 \mod 8$.

Then $[L] \in \mathbb{R}_1^f \cdot \mathbb{R}_n^f$.

In particular, exotic $S^8 \to X$.  

3. Floer theory

- \((\mathbb{R}^{2n}, \omega = dx)\) Liouville
- \(L, k \leq x\) closed exact Lag.s.

- Pick \(J\) an A Priori X

\[ M_{xy} := \left\{ u : [0, 1] \rightarrow \mathbb{R} \mid \begin{align*}
\delta_{\Sigma} u &= 0 \\
u(\mathbb{R} \times [0, 1]) &\leq k \\
u(\mathbb{R} \times 1) &\leq L \\
u(\mathbb{R} \times 0) &\leq k \\
u (+\infty) &= y \\
u (-\infty) &= x
\end{align*} \right\} \]

- Compactify to \(M_{xy}\)

\[ M_{xy} = M_{xy} \cup \text{broken strips} \]
Thin (Large): All $M_{xy}$ are compact smooth manifolds with corners with boundary:

$$\partial M_{xy} = \text{broken strips}$$

$$\bigcup \quad \text{z.e.L.N.K.} \quad M_{xz} \times M_{zy}$$

- If we assume $\oplus$, then all $M_{xy}$ are stably framed and compatible with breaking
Floor theory \( \mathbb{Z} \):

\[ CF^*(L, k; \mathbb{Z}) := \bigoplus_{x \in L \setminus k} \mathbb{Z} \cdot x. \]

Differential:

\[ \partial : x \in L \setminus k \]

\[ \mapsto x \# M_{xy} \neq 0 \quad \forall x \in L \setminus k \]

\[ := 0 \quad \text{if } M_{xy} \text{ has } \dim \neq 0. \]

\[ HF^*_*(L, k; \mathbb{Z}) := H^*_*(CF^*(L, k; \mathbb{Z}), \mathbb{Z}) \]

- Only used 0-Dim moduli spaces \( M_{xy} \).
Large (Strategy - Cohen-Jones-Segal)

Builds $\text{HF}(L, k; \mathbb{R}^\text{fr})$

a spectrum, whose homology is $\text{HF}_*(L, k; \mathbb{R})$.

Can treat this like a module over $\mathbb{R}^\text{fr}$

s.t.

\[
\text{HF}(L, k; \mathbb{R}^\text{fr}) \otimes \mathbb{R} = \text{HF}_*(L, k; \mathbb{R})
\]

Built using moduli spaces of all dim
Also builds product:

\[ HF(L, L'; \mathbb{R}^f) \boxtimes HF(L', L''; \mathbb{R}^f) \]

\[ \downarrow \]

\[ HF(L, L''; \mathbb{R}^f) \]

Homotopy associative.

(Spectral Donaldson-Fukaya cat)
Example. Suppose \( \Lambda \in \mathbb{Z}_\ell \). I have a non-empty model:

\[ M_{xy} \]

Has no \( \Theta \).

So \( M_{xy} \) is a closed, stably framed \( \text{mfld} \) of \( \text{Dim} = i \).

\[ [M_{xy}] \in \mathbb{Z}_i. \]

Moreover,

\[ \text{HF} (L, k; \mathbb{Z}^{fr}) = \text{Cone} \left( \mathbb{Z}^{fr} \xrightarrow{\beta} \mathbb{Z}^{fr} \right) \]

If \( i = n-1 \), then

\[ \text{HF}_* (L, k; \mathbb{Z}) \cong M_* (S^n). \]
In fact:

If $HF_*(L, K; \mathbb{Z}) \cong H_*(S)$

then

$HF(L, K, J^fr) \cong$

$\operatorname{cone}(J^fr \xrightarrow{\Theta} \mathbb{Z}^{\mathfrak{a}})^{fr}$

where $\Theta \in J^fr_{n-1}$
Lemma: If \( L, K \) satisfy \( \otimes \) and \( L, K \) have the same Floer theory over \( \mathbb{R}^{fr} \), then \( [L] = [K] \) in \( \mathbb{R}^{fr}_\mu \). (x)

Question: If \( L, K \) have some Floer theory \( \mathbb{Z} \), do they also \( \mathbb{Z}^{fr} \)?

No
Examples: Take $T^*S^n \neq T^*S^n$.

$\uparrow$ $\downarrow$

$\Leftrightarrow$

$M_{xy} = \emptyset$.

- $L = S^n$, pick $f: L \to \mathbb{R}$ Morse, 2 critical points.
- Let $k = L$ deformed by $f$.
- $M_{xy} = \text{Morse flows } x \mapsto x^y$. 

$\Leftrightarrow$
Assume $L,k$ have some Floer theory $\mathcal{F}$, and they're homology spheres.

$$HF_+(L,k;\mathbb{Z}) \cong \mathbb{Z}_+(S^3)$$

Can obstruction theory to turn an $\sim$ over $\mathbb{Z}$ into an $\sim$ over $\mathbb{Z}^{fr}$.

$: \Rightarrow$ Find an obstruction

$$\Theta \in \mathbb{Z}^{fr}_{n-1} \quad \delta \in \mathbb{Z}^{fr}_{n+2}$$

In example before, $\Theta = [\text{Max}]$
· If $\theta = 0$, $L, k$ do have same Floor theory $\Rightarrow$

$\Rightarrow [L] = [k] \in \mathbb{R}^+_f$

**Note:** Goal: $[L] - [k] \in \mathbb{R}^+_f$

**Obstruction:** $\Theta \in \mathbb{R}^+_\Omega$

**Strategy:** Kill $\Theta$

see what is left in $\mathbb{R}^+_n$

· Let $R = \bigg\{ \mathbb{R}^+_{n-1} \bigg\} \bigg( \mathbb{R}^+_{n-1} \bigg)$
Now consider $MF(...; R)$

$$:= HF(...; \mathbb{R}^+) \otimes R$$

- previous arguments work
- Can try some obstruction theory

$\Theta \in \mathbb{R}^{n-1}$ is 0

and $L, k$ have some Floer theory over $R$. 

Upshot:
\([L] - [k] \in R^n\) is 0.

\([L] - [k] \in \mathbb{R}^+_n\)

lies in

\[\ker \quad \text{Quotient} \quad \mathbb{R}^n \quad \mathbb{R}^n\]

"part of the ideal in \(\text{deg}_n\)"

\[CF(L, L ; \mathbb{R}^+_n) = \mathbb{R}^+_n(L)\]
Thm: If \( L \sim k / \mathbb{R} \), then \([L] = [k]\) in \( H_n(X)\).

\[
\begin{align*}
\text{CE}(L, k) &\otimes \text{CE}(k, L) \\
\oplus \text{CE}(L, L) &\otimes \text{CE}(k, k) \\
\text{CF}(L, L) &\otimes \text{CF}(k, k) \\
\text{CF}(L, k) &\oplus \text{CF}(k, L) \\
\text{CF}(L) &\oplus \text{CF}(k) \\
\text{C}_*(L) &\oplus \text{C}_*(k) \\
-\text{C}_*(k) &\oplus \text{C}_*(L) \\
\end{align*}
\]

\([L] - [k] = 0\)
\[ k_\circ(x) \rightarrow HH_*(F) \rightarrow H_*(X) \]