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Floer Theory & Framed Cobordism between exact Lagrangians

Based on joint work-in-progress with Ivan Smith

1. Background + questions

2. Framed bordism

3. Floer theory.

1. Background + Qs.

- (X^{2n}, ω) symplectic mfld.
- Today: X always Liouville
  - non-compact
  - $\omega = d\lambda$
  - nice @ ∞

Examples

- $X = T^*Q$
- $X = T^*Q_1 \# T^*Q_2 \# T^*Q_3$.
- smooth cx affine variety.

<u>Question</u>: What are the diffeo classes of closed exact Lags $L \subseteq X$

- $\operatorname{Dim} L = n = \frac{1}{2} \operatorname{Dim} X$

- $\lambda|_L \in \Omega^1(L)$ exact.

<u>Nearby Lag conjecture</u>

$\Rightarrow$ if $X = T^*Q$ then any such $L \subseteq X$ is diffeo to Q.

Thm (Abouzaid, Kragh, Guillermou)

If $X = T^*Q$

any closed exact $L \subseteq X$

is $\simeq Q$.

$\Rightarrow$ Same H_*, π_*

How different can $\simeq$ mflds be?

Example (Kervaire - Milnor)

- There are exactly 28 diffeo classes of oriented mfld which $\simeq S^7$.

- " 2 " $\simeq S^8$.

- All of these are homeomorphic.

2. Framed bordism

<u>Def</u>: A <u>stable framing</u> on a mfld M^n is an $\cong$ of v.b.s

$$TM \oplus \mathbb{R}^k_M \cong \mathbb{R}^{n+k}_M$$

· A <u>stably framed mfld</u> is a pair (M, ξ), where ξ is a stable framing on M.

· Given M, · there might be no ξ

· Or $\exists$ but not unique.

Examples

- $\emptyset$

- $S^n \hookrightarrow \mathbb{R}^{n+1}$

$$TS^n \oplus \mathbb{R}^1_{S^n} \cong \mathbb{R}^{n+1}_{S^n}$$

stable framing.

- S^1, S^3 with the Lie gp framing

$$TS^1 \cong \mathbb{R}^1$$
$$TS^3 \cong \mathbb{R}^3$$

NOT the same as

- If M has a stable framing, all p_i, $w_i(M) = 0$

$\Rightarrow \mathbb{R}P^2, \mathbb{C}P^2$ don't have.

<u>Def</u> A <u>framed bordism</u> (W, σ)

from (M^n, ξ) to (M', ξ')

is cobordism W from M to M'

and a stable framing σ

s.t. $\sigma|_M = \xi$

$$\sigma|_{M'} = -\xi'$$

- Framed bordism is an equiv relation

Def The framed bordism groups

$$\Omega_n^{fr} := \{\text{stably framed } n\text{-mflds}\} / \text{framed bordism}$$

- Abelian gp under $\amalg$
- Unit is $\emptyset$.
- Ω_*^{fr} is a graded ring

product is product of mflds.

n	Ω_n^{fr}
0	$\mathbb{Z}$
1	$\mathbb{Z}/2$
$\vdots$	$\vdots$
7	$\mathbb{Z}/240$
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

Generated by (S^1, Lie)

All 240 are $(S^7, \text{some framing})$

- Any exotic sphere admit a stable framing.
- Let Σ = the unique exotic S^8.
- $[\Sigma] \neq [S^8]$ in Ω_8^{fr}

for any stable framings.

- Let $(X^{2n}, \omega = d\lambda)$ be Liouville
- Let $L, K \subseteq X$ be 2 closed exact Lag.s

<u>Thm</u> (A.-Smith, in progress).

Assume
- L, K are homology spheres.
- L, K have the same Floer theory over $\mathbb{Z}$. ($\Leftrightarrow \simeq$ in Fukaya cat)

$(*)$
- TX stably trivial

$$\begin{array}{ccc} TX \oplus \mathbb{C}^k_x & \cong & \mathbb{C}^{n+k}_x \\ \uparrow & & \uparrow \\ TL \oplus \mathbb{R}^k & \cong & \mathbb{R}^{n+k} \end{array}$$

some k

Pick a $\cong$.

- L, K stably framed too

Then $[L]-[K] \in \Omega_n^{fr}$

is 2-torsion.

$$[L]-[K] \in \underbrace{\Omega_1^{fr}}_{\uparrow \mathbb{Z}/2} \cdot \Omega_{n-1}^{fr} \subseteq \Omega_n^{fr}$$

<u>Cor 1</u> (Abouzaid-Alvarez-Gavela Courte-Kragh)

If $L \subseteq T^*S^n$ closed exact Lag

Then $[L] \in \Omega_1^{fr} \cdot \Omega_{n-1}^{fr}$

$\Rightarrow$ If $n=8$, L diffeo to S^8.

Cor 2 Let $X = T^*S^n \# T^*S^n$,

and $L \subseteq X$ a closed exact Lag.

Assume:
- Maslov class of $L = 0$.
- $n \equiv 0, 4, 6, 7 \mod 8$.

Then $[L] \in \Omega_1^{fr} \cdot \Omega_{n-1}^{fr}$

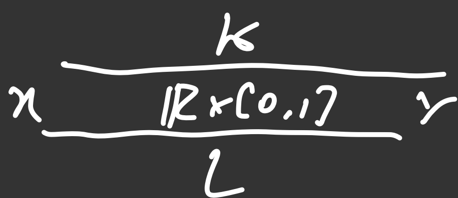
In particular, exotic $S^8 \not\hookrightarrow X$.

3. Floer theory

- $(X^{2n}, \omega = d\lambda)$ Liouville
- $L, K \subseteq X$ closed exact Lag.s.
- Pick J an ACS on X
- $\boxed{x, y \in L \cap K}$

Def $\mathcal{M}_{xy} := \{ u : \mathbb{R} \times [0,1] \to X \mid$
- $\bar{\partial}_J u = 0$
- $u(\mathbb{R} \times 0) \subseteq L$
- $u(\mathbb{R} \times 1) \subseteq K$
- $u(+\infty) = y$
- $u(-\infty) = x$

$\}$ / $\mathbb{R}$.



- Compactify to $\mathcal{M}_{xy}$

$= \mathcal{M}_{xy} \cup$ broken strips

<u>Thm</u> (Large):

- All $\mathcal{M}_{xy}$ are compact smooth mflds with corners with boundary:

$$\partial \mathcal{M}_{xy} = \text{broken strips} = \bigcup_{z \in L \cap K} \mathcal{M}_{xz} \times \mathcal{M}_{zy}$$

- If we assume $(*)$, then all $\mathcal{M}_{xy}$ are stably framed and compatible with breaking

Floer theory / $\mathbb{Z}$:

$$CF_*(L,K;\mathbb{Z}) := \bigoplus_{x \in L \cap K} \mathbb{Z}\cdot x$$

Differential:

$$\partial : x \in L \cap K \longmapsto \sum_{y \in L \cap K} \# \mathcal{M}_{xy} \cdot y$$

$:= 0$ if $\mathcal{M}_{xy}$ has Dim $\neq 0$.

$$HF_*(L,K;\mathbb{Z}) := H_*(CF_*(L,K;\mathbb{Z}), \partial)$$

- Only used 0-Dim moduli spaces $\mathcal{M}_{xy}$.

Large (Strategy — Cohen-Jones-Segal)

Builds $HF(L, K; \Omega^{fr})$

a spectrum, whose homology is

$HF_*(L, K; \mathbb{Z})$.

Can treat this like a

module over Ω^{fr}_*

s.t.

$$\widetilde{HF}(L, K; \Omega^{fr}) \otimes_{\Omega^{fr}} \mathbb{Z} = \widetilde{HF}_*(L, K; \mathbb{Z})$$

· Built using moduli spaces of all Dim

· Also builds product:

$$HF(L, L'; \Omega^{fr}) \otimes_{\Omega^{fr}} HF(L', L''; \Omega^{fr})$$

$$\downarrow$$

$$HF(L, L''; \Omega^{fr})$$

Homotopy associative.

(Spectral Donaldson-Fukaya cat).

Example Suppose $L \cap K = \{x, y\}$.

$\exists$ non-empty moduli

$\mathcal{M}_{xy}$.

Has no ∂

So $\mathcal{M}_{xy}$ is a closed, stably framed mfld, of Dim $= i$.

$[\mathcal{M}_{xy}] \in \Omega_i^{fr}$.

$\rightsquigarrow$

$HF(L, K; \Omega^{fr}) = \mathrm{Cone}\left(\Omega^{fr} \xrightarrow{\cdot [\mathcal{M}_{xy}]_i} \Sigma^i \Omega^{fr}\right)$

· If $i = n-1$, then

$HF_*(L, K; \mathbb{Z}) \cong H_*(S^n)$.

- In fact:

If $HF_*(L,K;\mathbb{Z}) \cong H_*(S^n)$

then

$HF(L,K;\Omega^{fr}) \simeq$

$$\mathrm{Cone}\left(\Omega^{fr} \xrightarrow{\cdot\Theta} \Sigma^{n-1}\Omega^{fr}\right)$$

where $\Theta \in \Omega^{fr}_{n-1}$

Lemma: If L, K satisfy ⊛ and L, K have the same Floer theory over Ω^{fr} then $[L] = [K]$ in $\Omega^{fr}_n(X)$

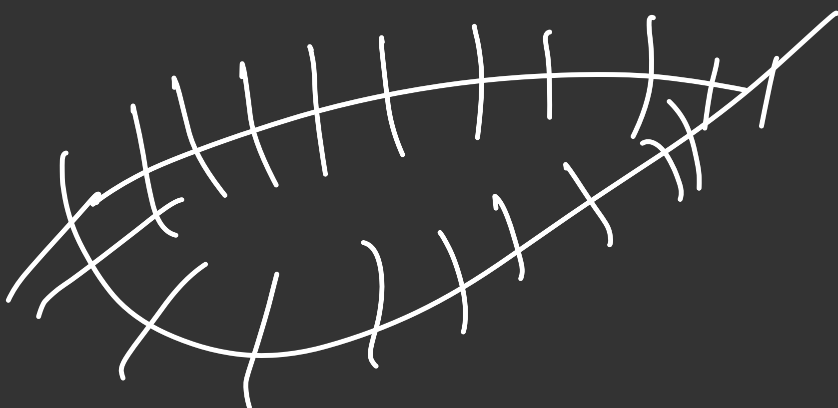
Question: If L, K have same Floer theory $/\mathbb{Z}$, do they also $/\Omega^{fr}$?

No

Examples: Take

$$T^*S^n \#_2 T^*S^n$$

$\uparrow$ L $\qquad$ $\uparrow$ K



$$\mathcal{M}_{xy} = \emptyset.$$

- $L = S^n$, pick $f : L \to \mathbb{R}$

morse, 2 crit points.

- Let $K = L$ deformed by f.
- $\mathcal{M}_{xy} =$ Morse flows $x \to y$, S^{n-1}

- Assume L, K have same Floer theory $/\mathbb{Z}$, and they're homology spheres.

$$\boxed{HF_*(L, K; \Omega^{fr}) \cong \underline{\Omega^{fr}_*(S^n)}}$$

- Can ~~try to do~~ obstruction theory to turn an $\cong$ over $\mathbb{Z}$ into an $\cong$ over Ω^{fr}.

$\rightsquigarrow$ Find an obstruction

$$\Theta \in \Omega^{fr}_{n-1} \qquad \delta \in \Omega^{fr}_{\sim n/2}$$

- In example before, $\Theta = [M_{xy}]$

- If $\Theta = 0$, L, K do have same Floer theory / Ω^{fr}
  $\Rightarrow [L] = [K]$ in Ω^{fr}

<u>Note</u>: Goal: $[L] - [K] \in \Omega^{fr}_n$

Obstruction: $\Theta \in \Omega^{fr}_{n-1}$

<u>Strategy</u>: kill Θ
see what is left in Ω^{fr}_n

- Let $R =$ " $\Omega^{fr} \big/ (\Omega^{fr}_{n-1})$ "

Now consider $HF(\ldots; R)$

$$:= HF(\ldots; \Lambda^{tr}) \otimes_{\Lambda^{tr}} R$$

- previous arguments work
- Can try same obstruction theory

$\Theta_R \in R_{n-1} = 0$ iso

$\rightsquigarrow$ L, K have same Floer theory over R.

Upshot:

$[L]-[K] \in R_n$ is 0.

$[L]-[K] \in \Omega^{fr}_n$

lies in

$$\ker\left(\Omega^{fr}_n \xrightarrow{\text{Quotient}} R_n\right) \ni \Omega^{fr}_{n-1} \cdot \Omega^{fr}_1$$

$$\shortparallel$$

"part of the ideal in deg n"

$$CF(L, L; \Omega^{fr}) = \Omega^{fr}_*(L)$$

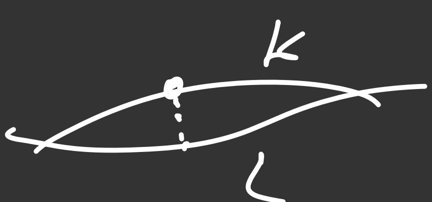
Thm: If $L \simeq K \ /\mathbb{R}$,

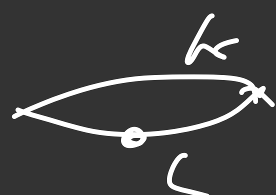
then $[L] = [K]$ in $H_n(X)$

$$
\begin{array}{ccc}
 & C_*(L) & \longmapsto [L] \\
 & \cong & \\
 & CF(L,L) & \\
CF(L,K) \otimes CF(K,L) \xrightarrow{\mu^2} & \oplus & \xrightarrow{\mathcal{OC}} C_*(X) \\
 & CF(K,K) & \longmapsto -[K] \\
 & \cong & \\
 & C_*(K) &
\end{array}
$$

$(1,1)$

$[L] - [K] = 0$





$$\begin{tikzcd}
K_0(X) \arrow[r] & MH_*(\mathcal{F}) \arrow[r] & H_*(X) \\
L \cong K \arrow[urr, bend right]
\end{tikzcd}$$