Surfaces of section, Anosov Reeb flows, and the C^2-stability conjecture for geodesic flows

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Joint work with Gonzalo Contreras

Surfaces of section

(N, λ) closed contact 3-manifold $\quad$ ($\lambda \wedge d\lambda$ nowhere zero)
X Reeb vector field $\quad$ ($d\lambda(X, \cdot) \equiv 0$, $\lambda(X) \equiv 1$)
$\phi_t : N \to N$ Reeb flow

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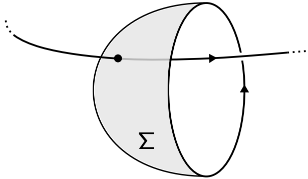
A surface of section is a compact immersed surface $\Sigma \looparrowright N$ such that:

- $\partial\Sigma$ is tangent to X,
- $\mathrm{int}(\Sigma)$ is embedded in $N \setminus \partial\Sigma$ and transverse to X,



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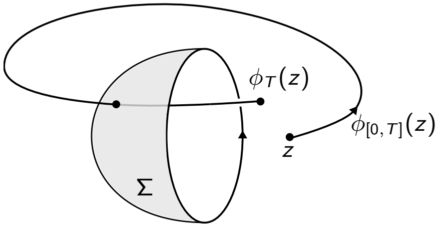
A global surface of section is a compact immersed surface $\Sigma \looparrowright N$ such that:

- $\partial\Sigma$ is tangent to X,
- $\mathrm{int}(\Sigma)$ is embedded in $N \setminus \partial\Sigma$ and transverse to X,
- for some $T > 0$, any orbit segment $\phi_{[0,T]}(z)$ intersects Σ.



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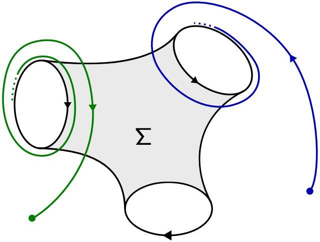
A almost global surface of section is a compact immersed surface $\Sigma \looparrowright N$ such that:

- $\partial\Sigma$ is tangent to X,
- $\mathrm{int}(\Sigma)$ is embedded in $N \setminus \partial\Sigma$ and transverse to X,
- Every orbit intersects Σ.
- Half-orbits not intersecting Σ in the future or in the past are asymptotic to a hyperbolic component of $\partial\Sigma$.



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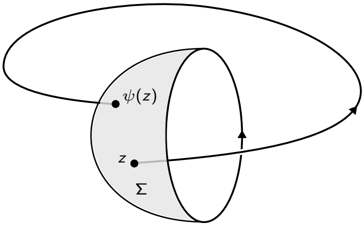
Return map: $\psi : U \to \mathrm{int}(\Sigma)$, $U \subset \mathrm{int}(\Sigma)$ open
$z \mapsto \phi_{\tau(z)}(z)$



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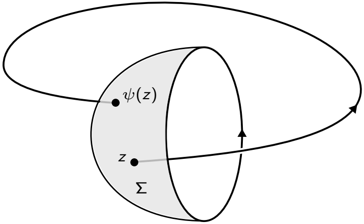

Remarks.

- (Poincaré recurrence) U has full measure in Σ.

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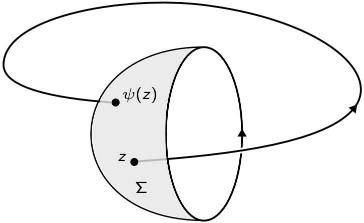

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Global surfaces of section – some history

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- (Hofer-Wysocky-Zehnder, 2003) Existence of global surfaces of sections for the Reeb flows of non-degenerate dynamically convex tight 3-spheres.

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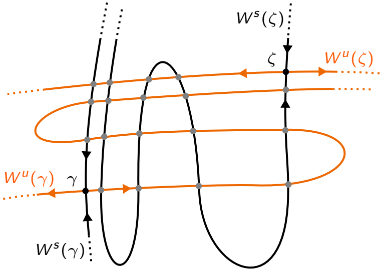

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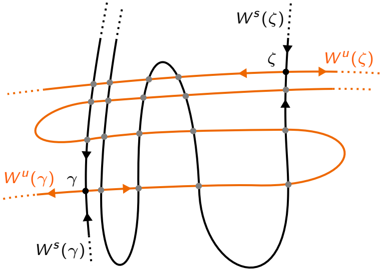

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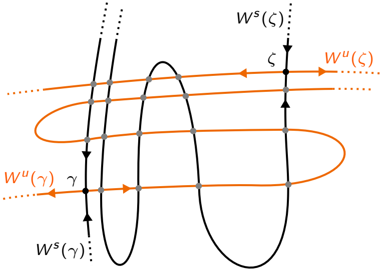

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Main theorem

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An alternative generic existence result:

Theorem (Colin-Dehornoy-Hryniewicz-Rechtman). *Any non-degenerate Reeb vector field on a closed 3-manifold with equidistributed closed orbits admits a global surface of section.*

Towards global surfaces of section

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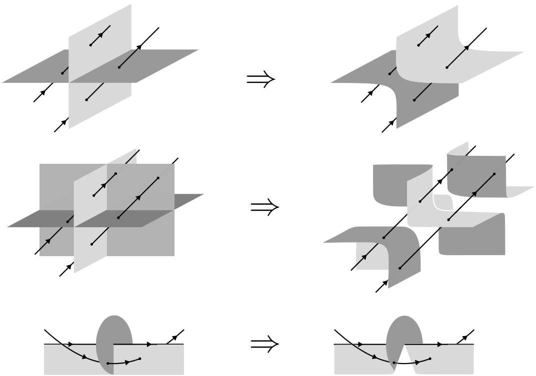
- (Colin-Dehornoy-Rechtman, 2020) Resolve self-intersections by means of Fried's surgery:



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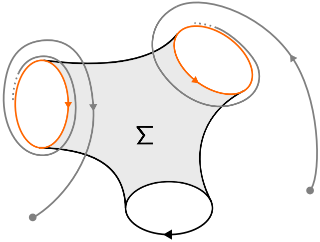

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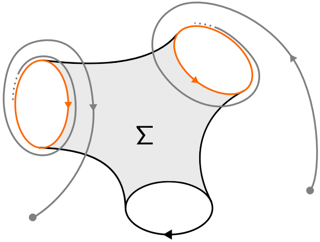

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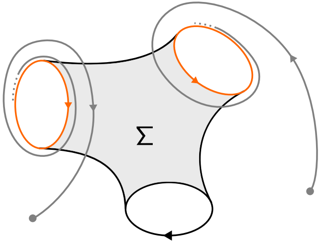

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Remark. Colin-Dehornoy-Rechtman employed such Σ to construct broken book decompositions of N.

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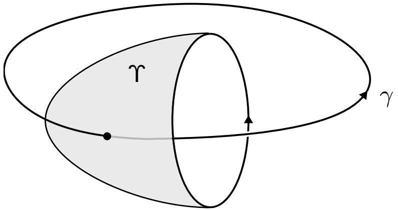
Lemma (Colin-Dehornoy-Rechtman) *If there exists a surface of section Υ such that $\partial\Upsilon \cap \partial\Sigma = \varnothing$ and $\mathrm{int}(\Upsilon) \cap \gamma \neq \varnothing$ for some $\gamma \subset \partial_\infty\Sigma$, then there exists an almost global surface of section $\Sigma' = \Sigma\#\Upsilon$ with $\partial_\infty\Sigma' = \partial_\infty\Sigma \setminus \{\gamma\}$.*



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Lemma (Colin-Dehornoy-Rechtman, Fried) *If there exists $\gamma \subset \partial_\infty \Sigma$ having transverse homoclinics in all separatrices, then there exists a surface of section Υ such that $\partial \Upsilon \cap \partial \Sigma = \varnothing$ and $\mathrm{int}(\Upsilon) \cap \gamma \neq \varnothing$.*

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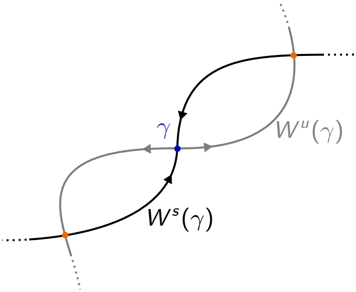

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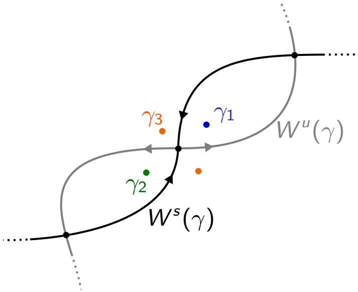

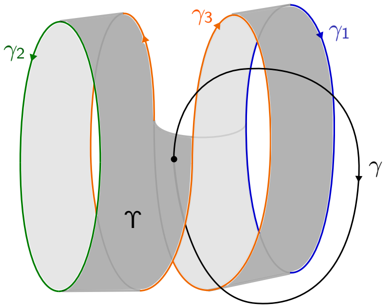

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Lemmas.

- *If there exists a surface of section* Υ *such that* $\partial\Upsilon \cap \partial\Sigma = \varnothing$ *and* $\mathrm{int}(\Upsilon) \cap \gamma \neq \varnothing$ *for some* $\gamma \subset \partial_\infty\Sigma$*, then there exists an almost global surface of section* $\Sigma' = \Sigma \# \Upsilon$ *with* $\partial_\infty\Sigma' = \partial_\infty\Sigma \setminus \{\gamma\}$.

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In order to find a global surface of section, we have to show that:

There is always some $\gamma \subset \partial_\infty\Sigma$ *with transverse homoclinics in all separatrices.*

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Theorem (Contreras-Mazzucchelli) *Any* $\gamma \subset \partial_\infty \Sigma$ *satisfies* $\overline{W^s(\gamma)} = \overline{W^u(\gamma)}$ *and has homoclinics in all separatrices.*

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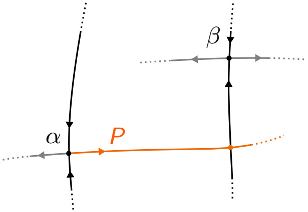

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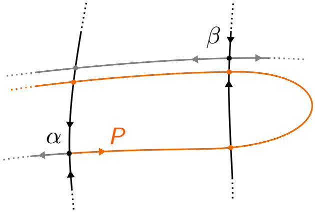

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From almost global to global

(N, λ) Kupka-Smale, $\Sigma \subset N$ almost global surface of section

Theorem (Contreras-Mazzucchelli) *Any $\gamma \subset \partial_\infty \Sigma$ satisfies $\overline{W^s(\gamma)} = \overline{W^u(\gamma)}$ and has homoclinics in all separatrices.*

Proof

- If $\gamma \subset \partial_\infty \Sigma$ has a homoclinic, then $\overline{W^s(\gamma)} = \overline{W^u(\gamma)}$.
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- (Hofer-Wysocki-Zehnder) For every $\gamma \subset \partial_\infty \Sigma$, every connected component $P \subset W^u(\gamma) \setminus \gamma$ satisfies $P \cap W^s(\partial_\infty \Sigma) \neq \varnothing$.
- $\gamma \subset \partial_\infty \Sigma$, and consider heteroclinic sequences among closed orbits in $\partial_\infty \Sigma$
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Applications

Application: Anosov Reeb flows

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X Reeb vector field
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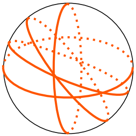

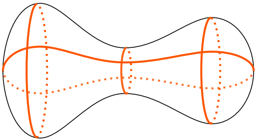

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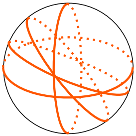
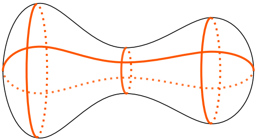
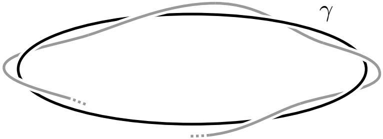




- Elliptic closed geodesic γ:



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Remark. For Hamiltonian flows, and in particular for Finsler geodesic flows, the analogous theorem was established by Newhouse.

A characterization of Anosov Reeb flows

Theorem (Contreras-Mazzucchelli). *Let* (N, λ) *be a closed contact 3-manifold such that:*

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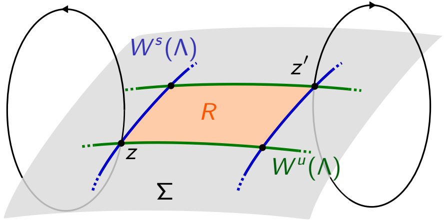
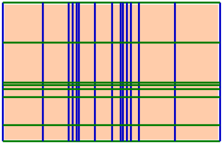
- We fix a small heteroclinic rectangle $R \subset \mathrm{int}(\Sigma)$:



$z, z' \in \mathrm{Per}(X) \cap \Lambda$

A characterization of Anosov Reeb flows

- $R \cap \left(W^s(\Lambda) \cup W^u(\Lambda) \right)$ is compact and connected



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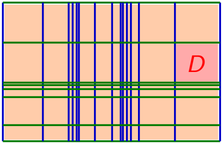

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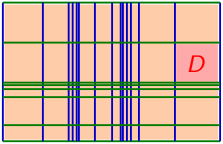

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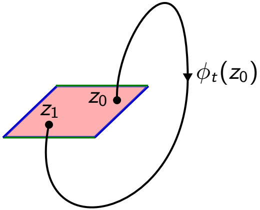

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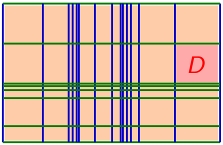

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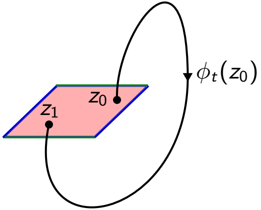

- Extend $z_0 \mapsto z_1$ to a smooth return map $\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma)$.

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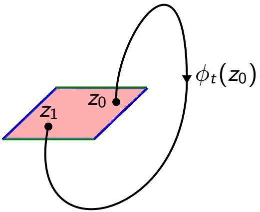

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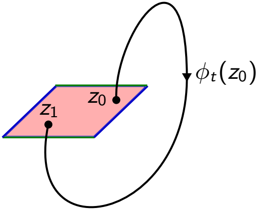

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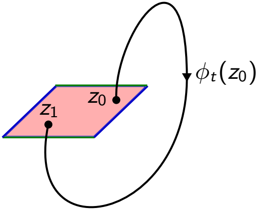

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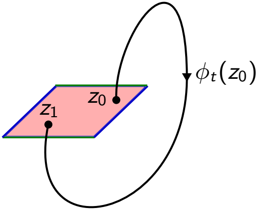

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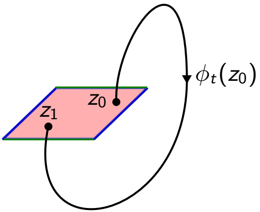

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- Thus $z \in D \cap \mathrm{Per}(X)$. But $D \cap \mathrm{Per}(X) \subset D \cap \Lambda = \varnothing$. □

Thank you for your attention!

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And if Σ is only an almost global return map?

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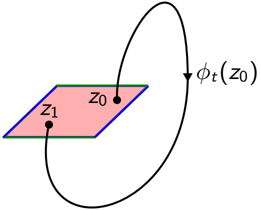

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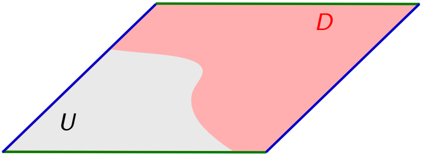

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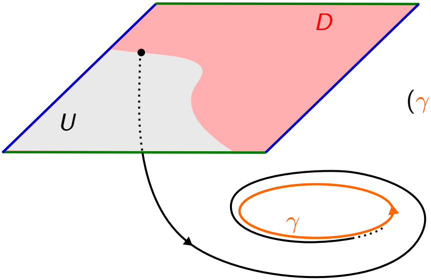

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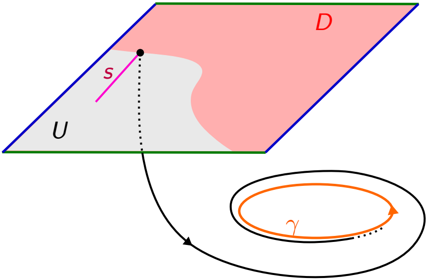

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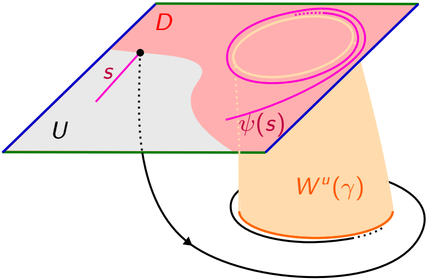

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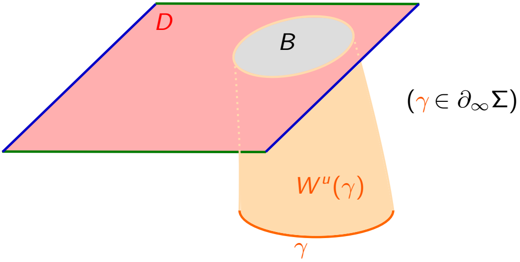

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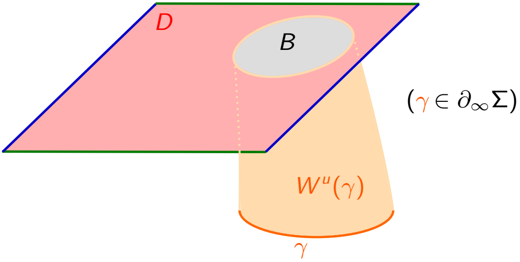
- Let's prove that $U = D$, if we chose D small enough!
- If not:



- $\mathrm{area}(D) = \int_D d\lambda \geq \int_B d\lambda = \int_\gamma \lambda \geq \min_{\zeta \subset \partial\Sigma} \int_\zeta \lambda.$ $\square$