Heaviness and relative symplectic cohomology.

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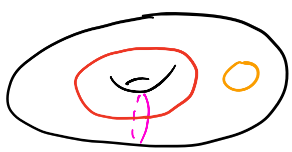
Fix a closed symplectic manifold (M, ω).

Symplectic rigidity:

A compact set K is called Hamiltonian non-displaceable if $\varphi(K) \cap K \neq \phi$, $\forall \varphi \in Ham(M, \omega)$.

Example:



(T^2, σ)

 not rigid

 rigid

 super rigid

Now we have several ways to describe this.

Different hierarchies of rigid sets:

	Rigid	Super-rigid
Spectral invariants	Heavy	Super-heavy
Relative SH	SH-visible	SH-full
Quantum measure	Non-trivial measure	Full measure
Fukaya Category	Non-trivial objects	Generating collection

1. Rigid sets are Hamiltonian non-displaceable.
2. Super-rigid sets are rigid, but not vice versa in general.
3. Any rigid set intersects super-rigid set.

I. Entov - Polterovich

II. Varolgunes, Tonkonog - Varolgunes, Borman - Sheridan - Varolgunes

III. Dickstein - Ganor - Polterovich - Zapolsky

IV. Fukaya, Donaldson, ...

Theorem (Albers, Biran-Cornea, EP, FOOO)

A (good) Lagrangian submanifold is heavy, if its self Floer cohomology is non-zero.

$K = L$

$HF(L) \neq 0$

Today we would like to introduce a similar criterion for general compact sets, by using relative symplectic cohomology.

Theorem (Mak-S-Varolgunes)

K is heavy if and only if $SH_M(K;\Lambda) \neq 0$.

(for any compact K in a closed symplectic manifold)

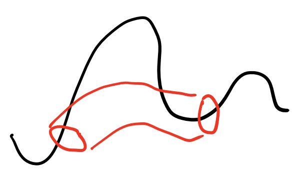
1. Spectral invariants.

Hamiltonian Floer theory on (M,ω).

Given a non-degenerate Hamiltonian function H, we form a chain complex $CF^*(H)$.

Generators: capped one-periodic orbits

Differentials: counting Floer cylinders.



Conventions: $\mathcal{A}_H([\gamma,u]) = \int_\gamma H + \int u^*\omega$

differentials increase action and degree.

PSS isomorphism: $QH^*(M) \longrightarrow HF^*(H)$

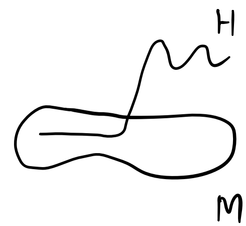
$$a \longmapsto PSS(a)$$

spectral invariants:

$$c(a;H) := \sup \{ \mathcal{A}_H(x) \mid x \in CF^*(H),\ dx = 0,\ [x] = PSS(a) \}.$$

A compact set K is called heavy, if

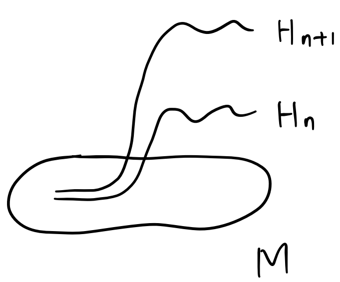
for any $H: M \longrightarrow \mathbb{R}_{\geq 0}$, $H|_K = 0$, $c(1;H) = 0$.



2. Relative symplectic cohomology.

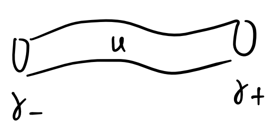
$$\Lambda = \{ \sum_{i=0}^{+\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R},\ \lambda_i < \lambda_{i+1} \longrightarrow +\infty \}.$$

$H_0 \leq \cdots \leq H_n \leq H_{n+1} \leq \cdots$ a monotone sequence



$H_n \longrightarrow 0$ on K

$H_n \longrightarrow +\infty$ on $M-K$



$\gamma_- \longmapsto \gamma_+ \cdot T^E$

$$E = \int u^*\omega + \int_{\gamma_+} H_+ - \int_{\gamma_-} H_- .$$

telescope $\quad tel(CF(H_n)) := \bigoplus_{n=1} \left(CF(H_n) \oplus CF(H_n)[1] \right)$

$$
\begin{tikzcd}
x_1 \in CF(H_1) \arrow[loop above, "d_1"] & x_2 \in CF(H_2) \arrow[loop above, "d_2"] & CF(H_3) \arrow[loop above, "d_3"] & \cdots \\
x_1' \in CF(H_1)[1] \arrow[u, "id"] \arrow[ur, "h_1"] \arrow[loop below, "d_1"] & x_2' \in CF(H_2)[1] \arrow[u, "id"] \arrow[ur, "h_2"] \arrow[loop below, "d_2"] & CF(H_3)[1] \arrow[u, "id"] \arrow[loop below, "d_3"] &
\end{tikzcd}
$$

$$\delta(x_1, x_1', x_2, x_2', \cdots) = \left(d_1 x + x_1', \ d_1 x_1', \ d_2 x_2 + h_1 x_1' + x_2', \ d_2 x_2', \cdots \right).$$

completion : allow infinite sums with valuations of $x_n, x_n' \longrightarrow +\infty$.

$$SH_M(K;\Lambda) := H\left(\widehat{tel}(CF(H_n)) \right).$$

Def. (Tonkonog – Varolgunes)

K is SH-visible if $SH_M(K;\Lambda) \neq 0$.

K is SH-full if $SH_M(K';\Lambda) = 0$ for any $K \cap K' = \phi$.

3. Applications.

Main strategy: use tools in the relative symplectic cohomology to study spectral invariants, and vice versa.

Relative symplectic cohomology	Spectral invariants.
Mayer-Vietoris sequence	huge literature
interpolation invariance	since 2000

Corollary 1. If K_1, K_2 are two disjoint non-heavy sets, then $K_1 \cup K_2$ is not heavy.

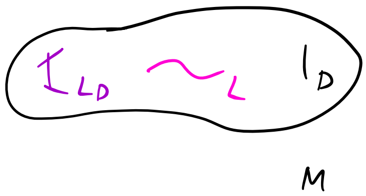
Proof.

$$\begin{tikzcd}
SH_M(K_1 \cup K_2; \Lambda) \arrow[rr] & & SH_M(K_1; \Lambda) \oplus SH_M(K_2; \Lambda) \arrow[dl] \\
& SH_M(K_1 \cap K_2; \Lambda) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

Corollary 2. Any Floer-essential Lagrangian L intersects the skeleton L_D in a Calabi-Yau manifold.

Proof. Floer-essential $\Rightarrow$ heavy $\Rightarrow$ $SH_M(L; \Lambda) \neq 0$.

On the other hand, Tonkonog-Varolgunes showed that L_D is SH-full.



Corollary 3. Any involutive map $\Phi : M \longrightarrow \mathbb{R}^N$ has at least one heavy fiber.

Proof. Varolgunes thesis showed that at least one fiber has non-vanishing relative SH.

4. Proofs.

Analyze the differentials in the completed telescope.

Consider a special element $(y_1, 0, 0, 0, \cdots)$, $y_1 \in CF(H_1)$, $d_1 y_1 = 0$.

then $\delta(y_1, 0, 0, 0, \cdots) = 0$

if $\delta(x_1, x_1', x_2, x_2', \cdots) = (y_1, 0, 0, 0, \cdots)$

then we have that

$$\begin{aligned} & d_1 x_1 + x_1' = y_1 && [y_1] = [x_1'] \text{ in } HF(H_1) \\ & d_1 x_1' = 0 && \\ & d_2 x_2 + h_1 x_1' + x_2' = 0 && [x_2'] = [h_1 x_1'] \text{ in } HF(H_2) \\ & d_2 x_2' = 0 && \vdots \\ & \vdots && \end{aligned}$$

on the othe hand, there is commutative diagram:

$$\begin{tikzcd}
HF(H_1) \arrow[r, "h_1"] & HF(H_2) \arrow[r, "h_2"] & HF(H_3) \arrow[r, "h_3"] & \cdots \\
QH(M) \arrow[u, "PSS^1"] \arrow[ur, "PSS^2"] \arrow[urr, "PSS^3"] & & \cdots &
\end{tikzcd}$$

$QH(M;\Lambda)$

$\Longrightarrow$ if y_1 represents $PSS^1(a)$

then x_1' represents $PSS^1(a)$,

x_2' represents $PSS^2(a)$,

$\vdots$

Proposition.

If $SH_M(K;\Lambda)=0$, then $\{c(a;H_n)\}_{n=1}^{+\infty}$ is unbounded.

($SH_M(K;\Lambda)=0 \Rightarrow K$ not heavy.)

The other direction needs a chain level product structure of $SH_M(K;\Lambda)$, recently constructed by Abouzaid - Groman - Varolgunes.

Abstract setting: suppose we have a chain complex equipped with a product structure in the chain level, satisfying the graded Leibniz rule, recovering the product in the homology. If the unit admits a chain level representative with positive valuation, then the unit vanishes in homology level.

Proof. $\tilde{x}$: a chain level representative of the unit and valuation > 0.

$$\tilde{x} * \tilde{x} - \tilde{x} = dy$$
$$\tilde{x}^3 - \tilde{x}^2 = \tilde{x} * dy = d(\tilde{x} * y)$$
$$\tilde{x}^4 - \tilde{x}^3 = \tilde{x} * \tilde{x} * dy = d(\tilde{x}^2 * y)$$
$$\vdots$$

formally $\tilde{x} = d(y + \tilde{x} * y + \tilde{x}^2 * y + \cdots)$

convergence : $\mathrm{val}(\tilde{x}) > 0 \implies \mathrm{val}(\tilde{x}^n * y) \longrightarrow +\infty$.

K not heavy $\implies$ $\exists\, x_1 \in CF(H_1)$, $d_1 x_1 = 0$, $[x_1] = PSS^1(1)$

and valuation of $x_1 > 0$

$\implies$ use x_1 to construct $\tilde{x}$

$\implies$ $SH_M(K;\Lambda) = 0$.