Symplectic Barriers

Symplectic Zoominar

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Lagrangian Intersection Phenomenon

Arnold’s Conjectures
Gromov’s P.H.C. Theory
Theory of Generating Functions
Floer Homology

"Lagrangian sub-manifolds have more intersection points than is required by topology."

On the other hand: “non-Lagrangians” are flexible as long as there are no topological obstructions.*
(Polterovich, Laudenbach-Sikorav, Gurel,....)

* Infinitesimally displaceable provided that the normal bundle has a nowhere-vanishing section.
Biran’s Lagrangian Barriers

Existence of irremovable intersections between contractible domains & Lagrangian submanifolds.

Biran’s decomposition theorem:

$$(M,\omega,J) = cw\text{-complex} \cong \text{skeleton } \Delta \text{\ symp disc bundle}$$

Theorem [Biran]: $P = (M,\omega,J,\Sigma)$ polarized Kähler of degree $k$.
For every $\phi : B(R) \hookrightarrow M$ with $R^2 \geq 1/\pi k$ one has $\phi(B(R)) \cap \Delta_P \neq \emptyset$.

Ex:

$$\begin{align*}
\phi(\mathbb{P}^n) &\supset \mathbb{R}P^n \\
\mathbb{R}P^n &\hookrightarrow \mathbb{R}P^n \\
\mathbb{Z}^n(R) &\hookrightarrow \mathbb{Z}^n(R) \\
\phi(B(R)) \cap \mathbb{R}P^n &\neq \emptyset.
\end{align*}$$
**Biran’s Lagrangian Barriers**

Lagrangian barriers are Lagrangian submanifolds such that:

\[ c_G(M \setminus L) < c_G(M). \]

\[ c_G(\mathbb{C}P^n \setminus \mathbb{R}P^n) = 1/2, \quad \text{(Biran)} \]

Other examples: Biran-Cornea (Clifford torus), Brendel-Schlenk (Markov pinwheels), Lee-Oh-Vianna (special Lagrangian tori), ...

**Theorem** [McDuff-Polterovich]: \( c_G(\mathbb{C}P^n \setminus \Gamma) = 1 \), where \( \Gamma \) is a complex submanifold.
“Lagrangian submanifolds are the most interesting submanifolds for several reasons.......these are the submanifolds that exhibit “symplectic rigidity” (Abbondandolo & Schlenk).

“Everything is a Lagrangian submanifold” (Weinstein)
The Existence of Symplectic Barriers

**Theorem** [Haim-Kislev, Hind, O]: \( \forall \delta > 0 \) there is a finite union of codimension two disjoint symplectic hyperplanes \( \Sigma \) such that every symplectic embedding \( B^{2n}(\delta) \xrightarrow{s} B^{2n}(1) \) must intersect \( \Sigma \). In other words, \( c_G(B^{2n}(1) \setminus \Sigma) < \pi \delta^2 \).

Main Point: Lagrangians are not the only source of symplectic rigidity. Non-Lagrangian submanifolds are not always “flexible”.
The Existence of Symplectic Barriers

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**Remarks:**

1) \( \forall \) union \( \tilde{\Sigma} \) of \( f \)-holomorphic planes \( c_G(B \setminus \tilde{\Sigma}) = c_G(B) \)

2) \( \exists \) "explicit construction" for \( \Sigma \)

3) After: Sackel-Song-Verolines-Zho

\[ c_G(B^4 \setminus V) = \?, \; V \in GV(4,2) \]
Sketch of the proof

\[ \Sigma_\epsilon = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid z_2 \in \mathbb{R}^2 \right\} \]

**Step I:** \( \forall \Omega \subseteq \mathbb{C}^2, \forall \gamma > 1 \)

\( \text{convex} \quad \forall \epsilon > 0 \)

\( \exists D \setminus N(\Sigma_\epsilon) \hookrightarrow \beta A^L D \)

"small" \( \mapsto \beta \epsilon \rightarrow 0 \)

(\text{Eliashberg})

**Step II:** \( \forall \gamma > 1 \exists \sigma \in \text{Sp}(2n) \)

\( s.t. \quad c(A^L \Sigma^{2n}) \ll \delta \)

The Thm follows for \( z^{-1} \Sigma_\epsilon \)
\[ \Sigma_\varepsilon = \{ (z_1, z_2) \in \mathbb{R}^2 \mid z_2 \in \varepsilon \mathbb{R}^2 \} \]

\( G_\alpha \) a dim sq vertices in \( \pi_2 \Sigma_\varepsilon \in \varepsilon \mathbb{R}^2 \)

\[ D_\varepsilon = \bigcup_\alpha \left( \left\{ \pi_1 x \mid x \in D \right\} \times \varepsilon \mathbb{R}^2 \right) \]
The Symplectic Embedding

\[ \mathbb{D}_\varepsilon \setminus N(\Sigma_\varepsilon) \xrightarrow{\text{symp}} \mathbb{A}^L \mathbb{D}_\varepsilon \]

\[ \text{ID} \times \psi \]

\[ \psi : \mathbb{R}^2 \setminus N(\varepsilon \mathbb{Z}^2) \rightarrow \mathbb{R}^2 \setminus \tilde{N}(\varepsilon L \mathbb{Z}^2) \]

Area Preserving !!!
Open Questions

1) \( C_G(B \setminus V) = ? \quad \forall \in Gr(4,2) \)

2) Understand "symplectic barriers" in terms of dynamics

3) \( \tilde{N}(3) : = \ast \sum_1 <B|Z(3)} \)

Estimate \( \tilde{N}(3) \)
THANKS FOR YOUR ATTENTION.

Any Questions?