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A RELATIVE CALABI–YAU STRUCTURE FOR LEGENDRIAN CONTACT HOMOLOGY

Symplectic Zoominar

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ABSTRACT

The duality long exact sequence relates linearised Legendrian contact homology and cohomology and was originally constructed by Sabloff in the case of Legendrian knots. We show how the duality long exact sequence can be generalised to a relative Calabi-+Yau structure, as defined by Brav and Dyckerhoff. We also discuss the generalised notion of the fundamental class and give applications. The structure is established through the acyclicity of a version of Rabinowitz Floer Homology for Legendrian submanifolds with coefficiens in the Chekanov-Eliashberg DGA. This is joint work in progress with Legout.

- **Part I:** Main result
- **Part II:** Based loop spaces and Absolute CY-structures
- **Part III:** Legendrian Invariants

  Contact Geometry and the Chekanov–Eliashberg algebra.
- **Part IV:** Relation to Sabloff duality and proof.

  Sabloff duality: Acylicity of Rabinowitz Floer Homology with $\Bbbk$ *coefficients.*

  Relative C–Y: Acylicity of Rabinowitz Floer Homology with DGA coefficients.
- **Part V:** The Fundamental class, and Applications

NOTATION

- The algebras here are typically differential graded algebras (DGAs) and the modules (resp. bimodules) are always DG-modules (resp. DG-bimodules).
- A DGA $\mathcal{A}_*$ has a differential ∂ of degree -1, i.e

$$\partial\colon \mathcal{A}_* \to \mathcal{A}_{*-1} = \mathcal{A}[-1]_*$$

- We write

$$M_* \xrightarrow{[k]} N_*$$

  to denote a map of degree k.

NOTATION

- If $\mathcal{A}$ is a DGA over $\Bbbk$, then an $\mathcal{A}$-bimodule is the same as a left

$$\mathcal{A}^e := \mathcal{A} \otimes_{\Bbbk} \mathcal{A}^{op}\text{–module.}$$

- "Left" and "right" can be confusing for bimodules, since $(\mathcal{A}^e)^{op} \cong \mathcal{A}^e$ by shuffling the order of the two factors of $\mathcal{A}^e = \mathcal{A} \otimes_{\Bbbk} \mathcal{A}^{op}$.
- The bimodule tensor product between a right bimodule M and a left bimodule N is consequently written as

$$M \otimes_{\mathcal{A}^e} N.$$

- The **inverse dualizing bimodule** of an $\mathcal{A}$-bimodule M is

$$M^!_* := R\mathrm{Hom}^*_{\mathcal{A}^e}(M, \mathcal{A}^e)$$

where $\mathcal{A}^e$ is a free $\mathcal{A}$-bimodule of rank one. Note that

$$(\mathcal{A}^e[k])^! = \mathcal{A}^e[-k].$$

- An $\mathcal{A}^e$-module morphism (i.e. bimodule morphism) between free bimodules

$$(\mathcal{A}^e)^{\oplus k} \xrightarrow{f} (\mathcal{A}^e)^{\oplus l}$$

corresponds to a $\mathcal{A}^e$-valued $k \times l$-matrix, and the adjoint map

$$(\mathcal{A}^e)^{\oplus k} \xleftarrow{f^!} (\mathcal{A}^e)^{\oplus l}$$

corresponds to the matrix obtained by taking
  - transpose and
  - reordering the factors of $\mathcal{A}^e$ in each entry.

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Part I: MAIN RESULT

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MAIN RESULT

- $\Lambda^n \subset \partial W^{2(n+1)}$ is a Legendrian, W is a subcritical Weinstein manifold;
- $\mathcal{A} = C_*\Omega(\Lambda)$ & $\mathcal{R} \twoheadrightarrow \mathcal{A}$ is a resolution of the diagonal $\mathcal{A}$-bimodule ($\mathcal{A}^e$-module);
- $\mathcal{C}_{\mathcal{A}}(\Lambda)$ is the Chekanov–Eliashberg $\mathcal{A}$-DGA of Λ, and

$$\mathcal{S} = \mathrm{Cone}\left(\overline{\mathcal{S}} \xrightarrow{[-1]} \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}\right) \twoheadrightarrow \mathcal{C}_{\mathcal{A}}$$

is a semi-free resolution of the diagonal $\mathcal{C}_{\mathcal{A}}$-bimodule ($\mathcal{C}^e_{\mathcal{A}}$-module).

$$\begin{array}{ccccccc}
\cdots\to & \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R} & \hookrightarrow & \mathcal{S}_* & \twoheadrightarrow & \overline{\mathcal{S}}_* & \overset{[-1]}{\cdots} \\
 & \simeq\Big\downarrow \mathcal{CY}_{\mathcal{A}} & & \simeq\Big\downarrow \mathcal{CY}_{\mathcal{C},\mathcal{A}} & & \simeq\Big\downarrow \overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}} & \\
\twoheadrightarrow & R\mathrm{Hom}^{*-n}_{\mathcal{C}^e_{\mathcal{A}}\otimes_{\mathcal{A}^e}\mathcal{R}}(\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}, \mathcal{C}^e_{\mathcal{A}}) & \overset{[-1]}{\dashrightarrow} & R\mathrm{Hom}^{*-n-1}_{\mathcal{C}^e_{\mathcal{A}}}(\overline{\mathcal{S}}, \mathcal{C}^e_{\mathcal{A}}) & \hookrightarrow & R\mathrm{Hom}^{*-n-1}_{\mathcal{C}^e_{\mathcal{A}}}(\mathcal{S}, \mathcal{C}^e_{\mathcal{A}}) & \text{—}
\end{array}$$

MAIN RESULT

- $\Lambda^n \subset \partial W^{2(n+1)}$ is a Legendrian, W is a subcritical Weinstein manifold;
- $\mathcal{A} = C_*\Omega(\Lambda)$ & $\mathcal{R} \twoheadrightarrow \mathcal{A}$ is a resolution of the diagonal $\mathcal{A}$-bimodule ($\mathcal{A}^e$-module);
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is a semi-free resolution of the diagonal $\mathcal{C}_{\mathcal{A}}$-bimodule ($\mathcal{C}^e_{\mathcal{A}}$-module).

$$\begin{array}{ccccccccc}
\cdots & \dashrightarrow & \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}_* & \xhookrightarrow{\iota} & \mathcal{S}_* & \overset{p}{\twoheadrightarrow} & \overline{\mathcal{S}}_* & \overset{[-1]}{\cdots\cdots} & \cdots \\
 & & \simeq \downarrow \mathcal{CY}_{\mathcal{A}} & & \simeq \downarrow \mathcal{CY}_{\mathcal{C},\mathcal{A}} & & \simeq \downarrow \overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}} & & \\
\cdots & \twoheadrightarrow & (\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R})^![-n] & \overset{[-1]}{\dashrightarrow} & \overline{\mathcal{S}}^![-n-1] & \xhookrightarrow{p^!} & \mathcal{S}^! & \overset{\iota^!}{\longrightarrow} & \cdots
\end{array}$$

$$M^! := R\mathrm{Hom}_{\mathcal{C}^e_{\mathcal{A}}}(M, \mathcal{C}^e_{\mathcal{A}})$$

MAIN RESULT

THEOREM (.–LEGOUT)

The above diagram induces a quasi-isomorphism between distinguished triangles. In other words, inclusion of unital DGAs

$$\mathcal{A} \hookrightarrow \mathcal{C}_{\mathcal{A}}$$

is **relative** $(n+1)$**-Calabi–Yau** in the sense of [BD19]; also see [KW22].

NEXT STEP

- The proof can (and will) be generalised to the case when Λ is a Legendrian embedding of a Weinstein skeleton, by using the DGA for singular Legendrians by Asplund–Ekholm [AE22] (in general dimensions) or An–Bae [AB20] (in dimension one).
- By the surgery formula [BEE12], [EL17], [Bä23], this endows the partially wrapped Fukaya category of an $2(n+1)$-dimensional Weinstein manifold with Weinstein stops with a relative $(n+1)$-Calabi–Yau structure.

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NEXT STEP

- $\Lambda^n \subset \partial W^{2(n+1)}$ is a Weinstein skeleton with critical handles attached along λ (link of Legendrian $n-1$-spheres), and W a <u>subcritical</u> Weinstein manifold;
- $\mathcal{A} = \mathcal{C}_{\mathbb{K}}(\lambda)$ & $\mathcal{R} \twoheadrightarrow \mathcal{A}$ is a resolution of the diagonal $\mathcal{A}$-bimodule ($\mathcal{A}^e$-module);
- $\mathcal{C}_{\mathcal{A}}(\Lambda)$ is the Chekanov–Eliashberg $\mathcal{A}$-DGA of the singular Legendrian Λ, and

$$\mathcal{S} = \mathrm{Cone}\left(\overline{\mathcal{S}} \xrightarrow{[-1]} \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}\right) \twoheadrightarrow \mathcal{C}_{\mathcal{A}}$$

a semi-free resolution of the diagonal $\mathcal{C}_{\mathcal{A}}$-bimodule ($\mathcal{C}^e_{\mathcal{A}}$-module).

$$\begin{array}{ccccccccc}
\cdots & \dashrightarrow & \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}_* & \xhookrightarrow{\iota} & \mathcal{S}_* & \xrightarrow{p} & \overline{\mathcal{S}}_* & \overset{[-1]}{\cdots\cdots} & \cdots \\
 & & \simeq \downarrow \mathcal{CY}_{\mathcal{A}} & & \simeq \downarrow \mathcal{CY}_{\mathcal{C},\mathcal{A}} & & \simeq \downarrow \overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}} & & \\
\cdots & \twoheadrightarrow & (\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R})^![-n] & \overset{[-1]}{\dashrightarrow} & \overline{\mathcal{S}}^![-n-1] & \xhookrightarrow{p^!} & \mathcal{S}^![-n-1] & \xrightarrow{\iota^!} & \cdots
\end{array}$$

Part II: BASED LOOP SPACES AND ABSOLUTE CALABI–YAU STRUCTURES

ABSOLUTE CALABI–YAU STRUCTURES

The quasi-isomorphism

$$\mathcal{CY}_{\mathcal{A}} \colon \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}_* \xrightarrow{\simeq} (\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R})^![-n] = R\mathrm{Hom}^{*-n}_{\mathcal{C}^e_{\mathcal{A}}}(\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}, \mathcal{C}^e_{\mathcal{A}})$$

comes from an *absolute* (weak smooth) n-CY structure on $\mathcal{A} = C_*\Omega(\Lambda)$ or $\mathcal{C}_{\mathbb{K}}(\lambda)$:
More precisely: Tensor the following by $\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \cdot$

$$\begin{array}{ccc} \mathcal{R}_* & \xrightarrow[\simeq]{\mathcal{CY}} & \mathcal{R}^![-n] = R\mathrm{Hom}^{*-n}_{\mathcal{A}^e}(\mathcal{R}, \mathcal{A}^e) \\ \Big\downarrow{\simeq} & & \Big\downarrow{\simeq} \\ \mathcal{A} & & \mathcal{A}^![-n] \end{array}$$

ABSOLUTE CALABI–YAU STRUCTURES

- The wrapped Fukaya category of an $2(n+1)$-dimensional Weinstein manifold W has an absolute (smooth weak) $(n+1)$-Calabi–Yau structure by Ganatra [Gan12].
- The absolute Calabi–Yau structure of $\mathcal{A} = \mathcal{C}_{\mathbb{K}}(\lambda)$, i.e. the Chekanov–Eliashberg DGA of a collection of Legendrian spheres, has recently also been established by Legout using SFT methods (soon to appear on arXiv).

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In the case of a based loop space of a closed manifold Λ, the Calabi–Yau structure on $\mathcal{A} = C_*\Omega(\Lambda)$ is topological:

$$\begin{array}{ccc} \mathcal{R}_* = C_*(\Lambda; C_*\Omega(\Lambda)^e) & \xrightarrow[\simeq]{\mathcal{CY}} & C^{n-*}(\Lambda; C_*\Omega(\Lambda)^e) = \mathcal{R}^![-n]_{-*} \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ C_*\Omega(\Lambda) & & C_*\Omega(\Lambda)^![-n] \end{array}$$

BASED LOOP SPACES

- The quasi-isomorphism

$$C_*(\Lambda; C_*\Omega(\Lambda)^e) \cong C^{n-*}(\Lambda; C_*\Omega(\Lambda)^e)$$

  can be interpreted as Poincaré duality for Λ with coefficients in the free rank-one bimodule $C_*\Omega(\Lambda)^e$.
- When Λ has boundary, the analogous Poincaré duality gives a relative Calabi–Yau structure.

$C_*(\Lambda; C_*\Omega(\Lambda)^e) \simeq C_*\Omega(\Lambda)$ can be seen from the following fibration:

$$\begin{array}{ccc} P(\Lambda) \times_\Lambda P(\Lambda) & \longrightarrow & P(\Lambda) \\ \downarrow & & \downarrow {\scriptstyle ev_1} \\ P(\Lambda) & \xrightarrow{ev_1} & \Lambda \end{array}$$

$$\Omega(\Lambda) \times \Omega(\Lambda) \hookrightarrow P(\Lambda) \times_\Lambda P(\Lambda) \twoheadrightarrow \Lambda$$

Note that $P(\Lambda) \times_\Lambda P(\Lambda) \cong \Omega(\Lambda)$ since any loop can be split into two paths (each going half way).

EXAMPLE: LAURENT POLYNOMIAL RINGS

The chains on the based loop space of

$$\mathbb{T}^n = (S^1)^n$$

has a particularly nice model

$$\mathcal{A} = \Bbbk[H_1(\mathbb{T}^n)] = \Bbbk[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] = H_0(C_*\Omega(\mathbb{T}^n)) \sim_{q.is} C_*\Omega(\mathbb{T}^n),$$

i.e. the Laurent polynomial ring with

- vanishing differential;
- all elements in degree zero.

EXAMPLE: LAURENT POLYNOMIAL RING

ONE-DIMENSIONAL KOSZUL RESOLUTION

$$\mathcal{R}_* = \mathrm{Cone}\left(\Bbbk[t^{\pm 1}]^e \cdot T \xrightarrow{f} \Bbbk[t^{\pm 1}]^e \cdot E\right) \xrightarrow{\mu} \Bbbk[t^{\pm 1}],$$

$$f(a \otimes_{\Bbbk} b \cdot T) = (at \otimes_{\Bbbk} b - a \otimes_{\Bbbk} tb) \cdot E,$$
$$\mu(a \otimes_{\Bbbk} b \cdot E) = ab$$

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EXAMPLE: LAURENT POLYNOMIAL RING

ONE-DIMENSIONAL KOSZUL RESOLUTION

The Koszul resolution of the diagonal bimodule

$$\mathcal{R}_* = \mathrm{Cone}\left(\mathbb{k}[t^{\pm 1}]^e \cdot T \xrightarrow{f} \mathbb{k}[t^{\pm 1}]^e \cdot E\right), \quad f((1 \otimes_{\mathbb{k}} 1) \cdot T) = (t \otimes_{\mathbb{k}} 1 - 1 \otimes_{\mathbb{k}} t) \cdot E$$

can be realised as the Morse homology of S^1 with $\mathbb{k}[t^{\pm 1}]^e$-coefficients.

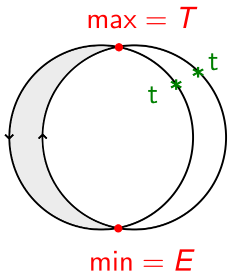

Figure 1: A Morse flowline on S^1 that contributes to $\partial((1 \otimes_{\mathbb{k}} 1) \cdot T) = (1 \otimes_{\mathbb{k}} 1 - t \otimes t^{-1}) \cdot E$.

EXAMPLE: LAURENT POLYNOMIAL RING

One-dimensional Koszul resolution

The Koszul resolution of the diagonal bimodule

$$\mathcal{R}_* = \text{Cone}\left(\Bbbk[t^{\pm 1}]^e \cdot T \xrightarrow{f} \Bbbk[t^{\pm 1}]^e \cdot E\right), \quad f((1 \otimes_{\Bbbk} 1) \cdot T) = (t \otimes_{\Bbbk} 1 - 1 \otimes_{\Bbbk} t) \cdot E$$

can be realised as the Morse homology of S^1 with $\Bbbk[t^{\pm 1}]^e$-coefficients.

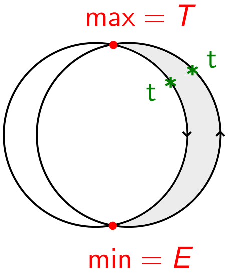

Figure 2: A Morse flowline on S^1 that contributes to $\partial((1 \otimes_{\Bbbk} 1) \cdot T) = (1 \otimes_{\Bbbk} 1 - t \otimes t^{-1}) \cdot E$.

EXAMPLE: LAURENT POLYNOMIAL RING

One can easily eatablish the absolute Calabi–Yau property by hand in this case:

ABSOLUTE (WEAK SMOOTH) 1-CALABI–YAU STRUCTURE

$$\mathrm{Cone}\left(\Bbbk[t^{\pm 1}]^e \cdot T \xrightarrow{f} \Bbbk[t^{\pm 1}]^e \cdot E\right)^! = \mathrm{Cone}\left(\Bbbk[t^{\pm 1}]^e \cdot T^! \xleftarrow{f^!} \Bbbk[t^{\pm 1}]^e \cdot E^!\right)$$

PROOF.

The map f corresponds to the symmetric 1×1-matrix $[t \otimes_{\Bbbk} 1 - 1 \otimes_{\Bbbk} t]$ □

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EXAMPLE: LAURENT POLYNOMIAL RING

For the Koszul resolution of Laurent polyonimal rings of n variables, we use the fact that

$$\Bbbk[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \Bbbk[t^{\pm 1}] \otimes_{\Bbbk} \dots \otimes_{\Bbbk} \Bbbk[t^{\pm 1}],$$
$$\Bbbk[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^e = \underbrace{\Bbbk[t^{\pm 1}]^e \otimes_{\Bbbk} \dots \otimes_{\Bbbk} \Bbbk[t^{\pm 1}]^e}_{n}$$

HIGH-DIMENSIONAL KOSZUL RESOLUTION

$$\mathcal{R} = \left(\text{Cone}\left(\Bbbk[t^{\pm 1}]^e \cdot T \xrightarrow{f} \Bbbk[t^{\pm 1}]^e \cdot E\right)\right)^{\otimes_{\Bbbk} n} \xrightarrow{\mu} \Bbbk[t_1^{\pm 1}, \dots, t_n^{\pm 1}],$$
$$\mu(a \otimes_{\Bbbk} b) \cdot E^n = ab$$

Part III: LEGENDRIAN INVARIANTS

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GEOMETRIC SETUP

- A **contact manifold** is a pair (Y^{2n+1}, ξ) where $\xi = \ker \alpha \subset TY$ is a field of tangent hyperplanes, satisfying the property that

$$\alpha \wedge d\alpha^{\wedge n} \in \Omega^{2n+1}(Y)$$

  is a volume form for any auxiliary choice of **contact form** $\alpha \in \Omega^1(Y)$.
- The choice of a contact form α induces a **Reeb vector field** $R_\alpha \in \Gamma(TY)$ defined by

$$\iota_{R_\alpha} = 1, \ \ \iota_{R_\alpha} d\alpha = 0.$$

- A **Legendrian submanifold** is an n-dimensional submanifold $\Lambda^n \subset Y^{2n+1}$ for which $T\Lambda \subset \xi$.

EXAMPLES

- The boundary of a Weinstein (or Liouville) domain

$$\partial(W, d\lambda) = (Y, \alpha = \lambda|_{TY})$$

  is a contact manifold.
- Today we consider the case when W is a subcritical Weinstein domain, i.e. whose completion is a symplectic product of the form $\overline{W} = \overline{V} \times \mathbb{C}$.

GEOMETRIC SETUP

MOST IMPORTANT EXAMPLE

The Darboux ball $W = B^{2(n+1)}$ with $\lambda = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)$ gives the standard fillable contact sphere

$$\left(S^{2n+1}, \alpha_{std} = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)\right)$$

Recall that

$$(S^{2n+1} \setminus \{pt\}, \ker \alpha_{std}) \cong \left(\mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}^n_{\mathbf{y}} \times \mathbb{R}_z, \ker \alpha_0\right)$$

where $\alpha_0 = dz - \sum_i y_i \, dx_i$; see [Gei08].

- As far as today's invariants are concerned, these contact manifolds are equivalent;
- The standard Reeb vector field $R_{\alpha_0} = \partial_z$ is particularly simple

THE STANDARD UNKNOT

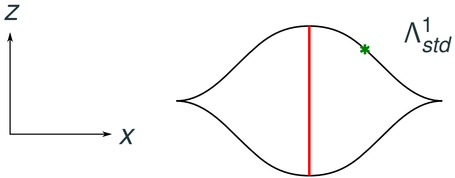

Figure 3: Front projection of the standard Legendrian unknot, where $y = \partial_x z$. There is a single Reeb chord.

THE HARVEY–LAWSON CONE

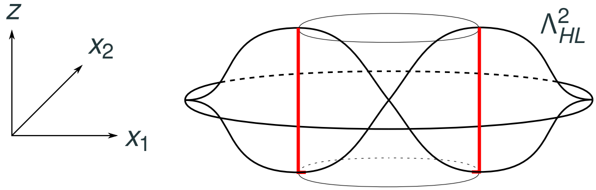

Figure 4: Front projection of the the Legendrian boundary of the Harvey–Lawson cone, where $y_i = \partial_{x_i} z$; see work [DRG19] by tue author and Golovko. There is a representative with precisely two Reeb chords.

Today we will investigate properties of the Chekanov–Eliashberg algebra

$$(\mathcal{C}_{\mathcal{A}}(\Lambda, \alpha), \partial), \;\; \mathcal{A} = C_*\Omega(\Lambda),$$

of a Legendrian submanifold $\Lambda \subset (Y, \alpha)$. This is an $\mathcal{A}$-DGA whose quasi-isomorphism class (even DG-homotopy class) is a strong invariant of the Legendrian isotopy class of Λ as considered e.g. by Ekholm–Lekili [EL17]. Also see work [BC07] by Barraud–Cornea for loop space coefficients in Floer homology.

Constructions (with simpler coefficients) are due to:

- Chekanov [Che02] and Eliashberg in the case $(Y, \alpha) = (\mathbb{R}^3, \alpha_0)$;
- Ekholm–Etnyre–Sullivan [EES07] in general contactisations $(P \times \mathbb{R}_z, dz - \lambda)$.
- Eliashberg–Hofer–Givental in the more general SFT-setting [EGH00].

The original version of the Chekanov–Eliashberg algebra has an underlying algebra

$$\mathcal{C}_{\mathbb{k}}(\Lambda) = \mathbb{k}\langle \mathcal{Q}(\Lambda) \rangle,$$

that is a fully non-commutative polynomial algebra over a field $\mathbb{k}$ with generators given by the Reeb chords $\mathcal{Q}(\Lambda)$ of Λ.

THE CHEKANOV–ELIASHBERG ALGEBRA

The more appropriate version today when Λ is a disconnected union of Legendrian spheres is

$$\mathcal{C}_{\mathbb{K}}(\Lambda) = \mathbb{K}\langle \mathcal{Q}(\Lambda) \rangle, \;\; \mathcal{Q}(\Lambda) \subset C \in \mathbb{K}^e - \text{mod},$$

with $\mathbb{K} = \Bbbk^{\pi_0(\Lambda)}$ semi-simple. (More details later.)

THEOREM 3.1 ([BEE12] IN HIGH DIM., [BÄ23] FOR SURFACES)

For an embedded Legendrian link $\Lambda \subset \partial W$ of spheres there is a quasi-isomorphism

$$\text{End}_{\mathcal{WF}uk(W_\Lambda)}(C_\Lambda, C_\Lambda) \xrightarrow{q.is.} \mathcal{C}_{\mathbb{K}}(\Lambda)$$

where C_Λ is the union of Lagrangian co-core discs of Weinstein handles attached along Λ.

WEINSTEIN HANDLE ATTACHMENT

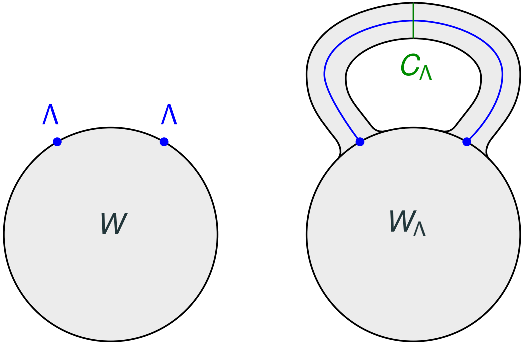

Figure 5: Weinstein surgery on a Legendrian sphere $\Lambda \subset \partial W$ produces the Weinstein manifold W_Λ, where C_Λ is the Lagrangian co-core disk of the Weinstein handle attachment.

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Today: We need a stronger version of the DGA

$$(\mathcal{C}_{\mathcal{A}}(\Lambda), \partial), \;\; \mathcal{A} = C_*\Omega(\Lambda),$$
$$\mathcal{C}_{\mathcal{A}} = \mathcal{A}\langle \mathcal{Q}(\Lambda) \rangle, \;\; \mathcal{Q}(\Lambda) \subset C \in \mathbb{K}^e - \text{mod}.$$

This version of the DGA has "loop space coefficients" in the DGA of chains of the based loop space of Λ.

Better terminology: $\mathcal{A} \hookrightarrow \mathcal{C}_{\mathcal{A}}$ is an $\mathcal{A}$-DGA; these "coefficients" are not central.

Important property: $\mathcal{A} \hookrightarrow \mathcal{C}_{\mathcal{A}}$ is a cofibrant, even *semi-projective*, $\mathcal{A}$-DGA.

A precise description of $\mathcal{C}_{\mathcal{A}}$:

First, recall that the underlying algebra of $\mathcal{C}_{\mathbb{k}}$, i.e. the non-commutative polyonimal ring, is a tensor algebra

$$\mathcal{C}_{\mathbb{k}} = \mathbb{k}\langle C_{\mathbb{k}}\rangle = \mathbb{k} \oplus C_{\mathbb{k}} \oplus C_{\mathbb{k}}^{\otimes_{\mathbb{k}} 2} \oplus C_{\mathbb{k}}^{\otimes_{\mathbb{k}} 3} \oplus \dots$$

generated by the free $\mathbb{k}$-module $C_{\mathbb{k}}$ with basis given by the Reeb chords on Λ.

A precise description of $\mathcal{C}_{\mathcal{A}}$:

Consider the ring of idempotents corresponding to components of Λ:

$$\mathbb{K} = \bigoplus_{i \in \pi_0(\Lambda)} e_i \Bbbk.$$

- $\mathcal{A}_* = C_*\Omega(\Lambda)$ is a $\mathbb{K}$-bimodule, and there is a splitting
$$C_*\Omega(\Lambda) = \bigoplus_i C_*\Omega(\Lambda_i) = \bigoplus_i e_i \cdot C_*\Omega(\Lambda) \cdot e_i,$$
into the loop space of the separate components of Λ.
- Let C be the projective $\mathbb{K}$-bimodule generated by the Reeb chords on Λ, s.t.
$$e_j \cdot a \cdot e_i = \begin{cases} a, & \Lambda_i \xrightarrow{a} \Lambda_j \\ 0, & \text{otherwise.} \end{cases}$$

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THE CHEKANOV–ELIASHBERG ALGEBRA

A precise description of $\mathcal{C}_{\mathcal{A}}$:

Consider the projective $\mathcal{A}$-bimodule

$$C_{\mathcal{A}} := \mathcal{A} \otimes_{\mathbb{K}} C \otimes_{\mathbb{K}} \mathcal{A} \subset \mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{k}} \mathcal{A}$$

generated by the Reeb chords.

The Chekanov–Eliashberg algebra conisdered here is an $\mathcal{A}$-cofibrant DGA with underlying algebra the tensor ring

$$\mathcal{C}_{\mathcal{A}} = \mathcal{A}\langle C_{\mathcal{A}} \rangle = \mathcal{A} \oplus C_{\mathcal{A}} \oplus C_{\mathcal{A}}^{\otimes_{\mathcal{A}} 2} \oplus C_{\mathcal{A}}^{\otimes_{\mathcal{A}} 3} \oplus \ldots$$

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- The differential of $\mathcal{C}_{\mathcal{A}}(\Lambda)$ is induced by the topological differential on $C_*\Omega(\Lambda)$ extended to all of $\mathcal{C}_{\mathcal{A}}(\Lambda)$ by counts of pseudoholomorphic discs and the graded Leibniz rule

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$$

for any $a, b \in \mathcal{C}_{\mathcal{A}}$.

THEOREM 3.2 ([EL17] IN HIGH DIM., [BÄ23] FOR SURFACES)

For any Legendrian submanifold $\Lambda \subset \partial W$ there is a quasi-isomorphism

$$\mathrm{End}_{\mathcal{WF}uk(W_\Lambda)}(C_\Lambda, C_\Lambda) \xrightarrow{q.is.} \mathcal{C}_{C_*\Omega(\Lambda)}(\Lambda)$$

where C_Λ is the union of Lagrangian linking discs that corresponds to components of the stop Λ in the Weinstein sector W_Λ.

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WEINSTEIN HALF-HANDLE ATTACHMENT

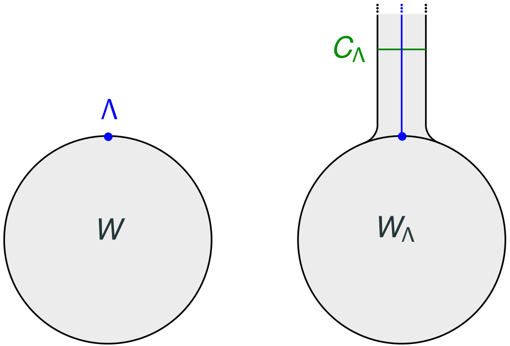
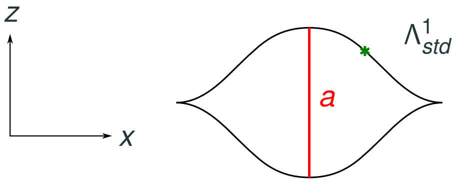


Figure 6: Attaching a Weinstein half-handle along a Legendrian Λ produces the Weinstein sector W_Λ, where C_Λ is the corresponding Lagrangian linking disc.

THE STANDARD UNKNOT

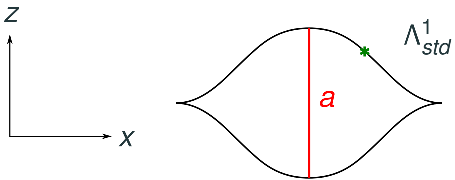


The differential with $\mathbb{k}$-coefficients is:

$$\partial a = 0,$$
$$|a| = 1.$$

The DGA is a polynomial algebra of a variable *a* in degree one.

THE STANDARD UNKNOT



The differential with $\Bbbk[t^{\pm 1}] \sim C_*\Omega(S^1)$-coefficients is:

$$\partial a = 1 - t^{-1},$$
$$|a| = 1.$$

PROPOSITION 3.3 (SEE [BCL18] OR [BÄ23])

$$\mathcal{A}_{\Bbbk[t^{\pm 1}]}(\Lambda^1_{Hopf}) \xrightarrow{q.is} H_0\left(\mathcal{A}_{\Bbbk[t^{\pm 1}]}(\Lambda^1_{std})\right) = \Bbbk$$

THE HOPF LINK

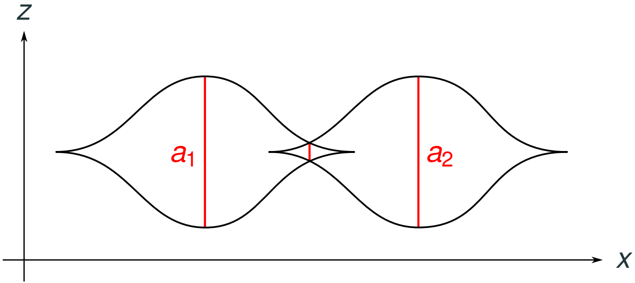


Figure 7: The front projection of the Hopf link Λ^1_{Hopf} in a Darboux chart.

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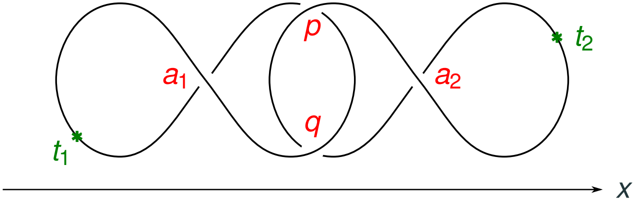

Figure 8: The Lagrangian projection (to the *xy*-plane) of the Hopf link in a Darboux chart.

The differential with coefficients in

$$C_*\Omega(S^1 \sqcup S^1) \sim \mathbb{k}[t_1^{\pm 1}] \oplus \mathbb{k}[t_2^{\pm 1}] :$$

$$\partial a_1 = e_1 - t_1 + pq, \;\; \partial a_2 = e_2 - t_2 + qp$$
$$|p| = 0 = |q|, \;\; |a_i| = 1.$$

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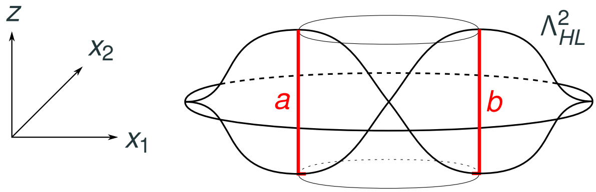

The differential with coefficients in $C_*\Omega(\mathbb{T}^2) \sim \Bbbk[t_1^{\pm 1}, t_2^{\pm 1}]$ was computed in [DRG19]:

$$\partial a = t_1 b - b t_1, \ \ \partial b = 1 + t_1 + t_2,$$

$$|a| = 2, \ \ |b| = 1.$$

PROPOSITION 3.4 (D.R.–GHIGGINI)

$$\mathcal{A}_{\Bbbk[t_1^{\pm 1}, t_2^{\pm 1}]}(\Lambda^2_{HL}) \xrightarrow{q.is.} H_0(\mathcal{A}_{\Bbbk[t_1^{\pm 1}, t_2^{\pm 1}]}(\Lambda^2_{HL})) = \Bbbk[t_1^{\pm 1}, t_2^{\pm 1}]/\langle 1 + t_1 + t_2 \rangle$$

(i.e. the ring of regular functions on the 1-dim. pair of pants, which also is 1-CY).

Part IV: RELATION TO SABLOFF DUALITY AND THE PROOF

SABLOFF DUALITY

- In [Sab06] Sabloff discovered a remarkable type of Poincaré duality for the "linearised" Legendrian contact homology (LCH) of a Legendrian. (Linearised LCH is an invariant derived entirely from the DGA.)
- In [EES09] Ekholm–Etnyre–Sabloff generalised this to a "Duality long exact sequence" that holds for the linearised LCH of horizontally displaceable Legendrians.
- The duality LES was generalised to the "bilinearised" LCH complex $\mathrm{Hom}^*_{Aug_-}(\varepsilon_0, \varepsilon_1)$ by Bourgeois–Chantraine in [BC14].
- When Λ is disconnected, we consider a version of $\mathrm{Hom}^*_{Aug_-}(\varepsilon_0, \varepsilon_1)$ that corresponds to finite-dimensional representations of the Fukaya category. There are different versions!

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SABLOFF DUALITY

Let $\Lambda \subset \partial B^{2(n+1)}$ be a closed Legendrian (being contained in the boundary of a subcritical Weinstein manifold should be sufficient) and a pair of $\mathbb{K}$-augmentations

$$\varepsilon_i \colon \ (\mathcal{C}_{C_*\Omega(\Lambda)}(\Lambda), \partial) \to (\mathrm{End}_{\Bbbk}(\mathbb{K}), 0), \ \ i = 0, 1,$$

of the Chekanov–Eliashberg algebra the following holds:

THEOREM (THE DUALITY LONG EXACT SEQUENCE [EES09])

$$\cdots \xrightarrow{[-1]} H_{*-1}(\Lambda; \nabla_0 \otimes_{\Bbbk} \nabla_1^{-1}) \longrightarrow H\mathrm{Hom}^{n-*+1}_{Aug_-}(\varepsilon_0, \varepsilon_1) \longrightarrow H\mathrm{Hom}^{*}_{Aug_-}(\varepsilon_1, \varepsilon_0)^{\vee} \xrightarrow{[-1]} \cdots$$

Where

$$\nabla_i = \left[\varepsilon_i|_{C_0\Omega(\Lambda)}\right] : \Bbbk[\pi_1(\Lambda)] \to \Bbbk$$

is the $\Bbbk$-local system on Λ induced by ε_i.

DEFINITION OF BILINEARISED LCH

There is an equivalence between:

- Morphisms of $\mathbb{K}$-DGAs (i.e. idempotents are respected):

$$\varepsilon_i \colon (\mathcal{C}_{C_*\Omega(\Lambda)}(\Lambda), \partial) \to (\mathrm{End}_{\Bbbk}(\mathbb{K}), 0), \quad i = 0, 1,$$

- Left $\mathcal{C}_{\mathcal{A}}$-modules $\mathbb{K}_{\varepsilon_i}$ with underlying vector space $\mathbb{K}$ that when restricted to $\mathbb{K} \subset \mathcal{C}_{\mathcal{A}}$ gives the canonical $\mathbb{K}$-module structure.

Recall that $M \rightsquigarrow M^\vee := \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ interchanges left and right modules.

DEFINITION OF BILINEARISED LCH [BC14]

The underlying left $\mathcal{C}^e_{\mathcal{A}}$-module: (differential is defined by left multiplication)

$$\mathrm{Hom}^*_{Aug_-}(\varepsilon_0, \varepsilon_1) = \mathbb{K}_{\varepsilon_1} \otimes_{\mathbb{K}} C^* \otimes_{\mathbb{K}} \mathbb{K}^\vee_{\varepsilon_0} \subset C^* \otimes_{\Bbbk} \mathrm{Hom}_{\Bbbk}(\mathbb{K}_{\varepsilon_0}, \mathbb{K}_{\varepsilon_1}),$$

where C is the $\mathbb{K}$-bimodule of Reeb chords.

Definition of bilinearised LCH [BC14]

The underlying left $\mathcal{C}^e_{\mathcal{A}}$-module: (differential is defined by left multiplication)

$$\mathrm{Hom}^*_{Aug_-}(\varepsilon_0, \varepsilon_1) = \mathbb{K}_{\varepsilon_1} \otimes_{\mathbb{K}} C^* \otimes_{\mathbb{K}} \mathbb{K}^\vee_{\varepsilon_0} \subset C^* \otimes_{\Bbbk} \mathrm{Hom}_{\Bbbk}(\mathbb{K}_{\varepsilon_0}, \mathbb{K}_{\varepsilon_1}),$$

where C is the $\mathbb{K}$-bimodule of Reeb chords.

The differential is the adjoint $\partial^\vee_{\varepsilon_0,\varepsilon_1}$, where:

$$\partial_{\varepsilon_0,\varepsilon_1}(\hat{c}) = \sum_i \sum_j \varepsilon_0(r_{i,1} b_{i,1} \cdots r_{i,j}) \hat{b}_{i,j} \varepsilon_1(r_{i,j+1} b_{i,j+1} \cdots r_{i,m_i+1}) \in \mathbb{K}^\vee_{\varepsilon_0} \otimes_{\mathbb{K}} \otimes C \otimes_{\mathbb{K}} \mathbb{K}_{\varepsilon_1}$$

is determined by the Chekanov–Eliashberg differential of the chord c

$$\partial_{\mathcal{C}}(c) = \sum_i r_{i,1} b_{i,1} r_{i,2} \ldots b_{i,m_i} r_{i,m_i+1}, \quad r_{i,j} \in \mathcal{A}, b_{i,j} \in C.$$

DEFINITION OF BILINEARISED LCH

DEFINITION OF $+$-VERSION OF BILINEARISED LCH FROM [NRS+20],[EL17]

The underlying left $\mathcal{C}^e_{\mathcal{A}}$-module: (differential is defined by left multiplication)

$$\mathrm{Hom}^*_{Aug_+}(\varepsilon_0, \varepsilon_1) = (\mathbb{K}_{\varepsilon_1} \otimes_{\mathbb{K}} C^* \otimes_{\mathbb{K}} \mathbb{K}^\vee_{\varepsilon_0}) \oplus C^*_{Morse}(\Lambda, \mathbb{K}),$$

where C and $C^*_{Morse}(\Lambda, \mathbb{K})$ are $\mathbb{K}$-bimodules of Reeb chords and critical points of a Morse function on Λ.

There is a cone structure

$$\mathrm{Hom}^*_{Aug_+}(\varepsilon_0, \varepsilon_1) = \mathrm{Cone}\left(C^*_{Morse}(\Lambda, \mathbb{K}) \xrightarrow{[1]} \mathrm{Hom}^*_{Aug_-}(\varepsilon_0, \varepsilon_1)\right)$$

that computes

$$R\mathrm{Hom}_{\mathcal{C}^e_{\mathcal{A}}}(\mathcal{C}^e_{\mathcal{A}}, \mathbb{K}_{\varepsilon_1} \otimes_{\Bbbk} \mathbb{K}^\vee_{\varepsilon_0})$$

see Ekholm–Lekili's work [EL17].

Proof of Sabloff Duality:

Start with the following canonical short exact sequence that arises from the cone structure of $\mathrm{Hom}^*_{Aug_+}$:

$$
\begin{array}{ccccc}
 & C_{*-1}(\Lambda; \nabla_0 \otimes_{\Bbbk} \nabla_1^{-1}) & \hookrightarrow & \mathrm{Hom}^*_{Aug_+}(\varepsilon_1, \varepsilon_0)^\vee & \twoheadrightarrow \mathrm{Hom}^*_{Aug_-}(\varepsilon_1, \varepsilon_0)^\vee \\
 & \downarrow \mathrm{PD} & & & \\
\cdots \twoheadrightarrow & C^{n-*+1}(\Lambda; \nabla_0 \otimes_{\Bbbk} \nabla_1^{-1}) & & \mathrm{Hom}^{n-*+1}_{Aug_-}(\varepsilon_0, \varepsilon_1) & \hookrightarrow \mathrm{Hom}^{n-*+1}_{Aug_+}(\varepsilon_0, \varepsilon_1) \longrightarrow \cdots
\end{array}
$$

Proof of Sabloff Duality:

WHEN Λ IS "HORIZONTALLY DISPLACEABLE":

$$\begin{array}{ccccc} & C_{*-1}(\Lambda; \nabla_0 \otimes_{\Bbbk} \nabla_1^{-1}) & \hookrightarrow & \mathrm{Hom}^*_{Aug_+}(\varepsilon_1, \varepsilon_0)^\vee & \twoheadrightarrow & \mathrm{Hom}^*_{Aug_-}(\varepsilon_1, \varepsilon_0)^\vee \\ & \downarrow \text{PD} & & \downarrow q.is & & \downarrow q.is \\ \cdots \twoheadrightarrow & C^{n-*+1}(\Lambda; \nabla_0 \otimes_{\Bbbk} \nabla_1^{-1}) & & \mathrm{Hom}^{n-*+1}_{Aug_-}(\varepsilon_0, \varepsilon_1) & \hookrightarrow & \mathrm{Hom}^{n-*+1}_{Aug_+}(\varepsilon_0, \varepsilon_1) & \longrightarrow \cdots \end{array}$$

OBS: The DGA $\mathcal{C}_{\mathcal{A}}(\Lambda)$, $\mathcal{A} = C_*\Omega(\Lambda)$, contains all information in the above diagram except for the vertical arrows.

How to prove the duality LES:

WHEN Λ IS "HORIZONTALLY DISPLACEABLE":

$$\begin{array}{ccccc} & C_{*-1}(\Lambda;\nabla_0\otimes_{\Bbbk}\nabla_1^{-1}) & \hookrightarrow & \mathrm{Hom}^*_{Aug_+}(\varepsilon_1,\varepsilon_0)^\vee & \twoheadrightarrow & \mathrm{Hom}^*_{Aug_-}(\varepsilon_1,\varepsilon_0)^\vee \\ & \downarrow \text{PD} & & \downarrow q.is & & \downarrow q.is \\ \cdots \twoheadrightarrow & C^{n-*+1}(\Lambda;\nabla_0\otimes_{\Bbbk}\nabla_1^{-1}) & & \mathrm{Hom}^{n-*+1}_{Aug_-}(\varepsilon_0,\varepsilon_1) & \hookrightarrow & \mathrm{Hom}^{n-*+1}_{Aug_+}(\varepsilon_0,\varepsilon_1) \longrightarrow \cdots \end{array}$$

OBS: The cones over each vertical arrows are versions of the Rabinowitz–Floer complex of Λ; this is acyclic in the horizontally displaceable case.

RABINOWITZ FLOER COMPLEX WITH DGA COEFFICIENTS

The LCH-version of the Rabinowitz Floer complex was constructed by Legout in [Leg20]. **For our purposes, we need a version with $\mathcal{C}^e_{\mathcal{A}}$-coefficients.**

Let Λ_1 be a small push-off of $\Lambda_0 = \Lambda$ constructed by a perturbation of the negative Reeb flow applied to Λ_0.

THE RABINOWITZ–FLOER COMPLEX

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-$$

This is a complex given as iterated cone of projective left $\mathcal{C}^e_{\mathcal{A}}$-modules generated by Reeb chords with one endpoint on each Λ_i, $i = 0, 1$.

RABINOWITZ–FLOER COMPLEX WITH DGA COEFFICIENTS

The Rabinowitz–Floer Complex

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-$$

This is a complex given as iterated cone of projective left $\mathcal{C}^e_{\mathcal{A}}$-modules generated by Reeb chords with one endpoint on each Λ_i, $i = 0, 1$.

- C^+ is the $\mathcal{C}^e_{\mathcal{A}}$-module of long Reeb chords from Λ_1 to Λ_0 (they correspond to Reeb chords on Λ);
- C^0 is the $\mathcal{C}^e_{\mathcal{A}}$-module of short Reeb chords from Λ_1 to Λ_0 (they correspond to critical points of a Morse function on Λ);
- C^- is the $\mathcal{C}^e_{\mathcal{A}}$-module of Reeb chords from Λ_0 to Λ_1 (they correspond to Reeb chords on Λ);

The correspondence of Reeb chords:

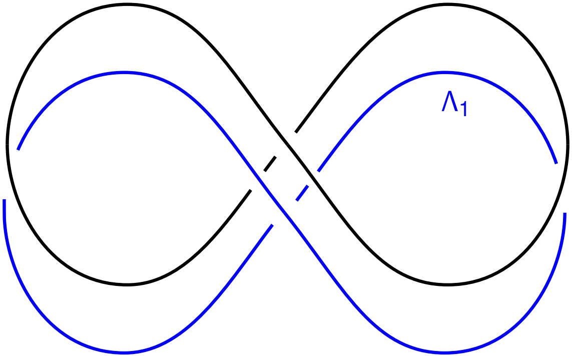


Figure 9: The Legendrian Λ_0 and a push-off Λ_1 by a perturbation of its image under the positive Reeb flow. (This shows the Lagrangian projection.)

The correspondence of Reeb chords:

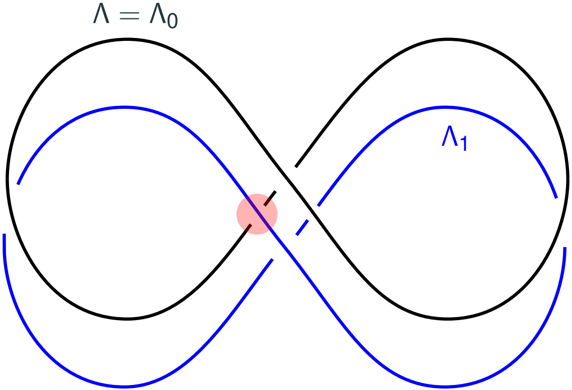


Figure 10: A long chord in C^- from Λ_0 to Λ_1 that corresponds to a chord on Λ.

The correspondence of Reeb chords:

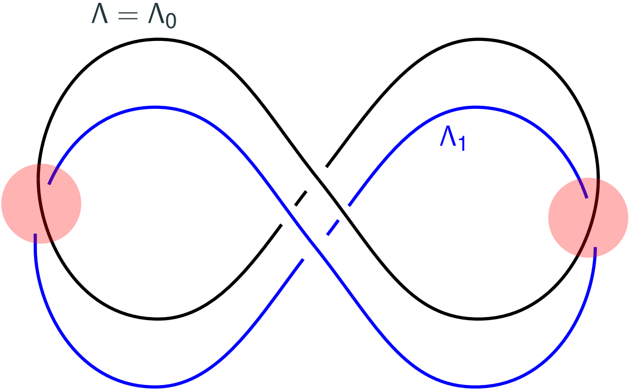


Figure 11: Short chord in C^0 from Λ_1 to Λ_0 that corresponds to a critical points of a Morse function on Λ.

The correspondence of Reeb chords:

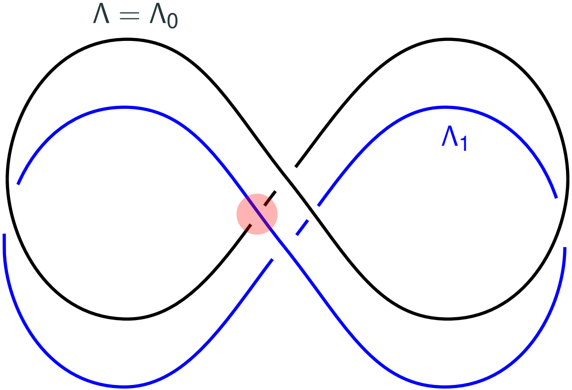


Figure 12: A long chord in C^+ from Λ_1 to Λ_0 that corresponds to a chord on Λ.

RABINOWITZ–FLOER COMPLEX WITH DGA COEFFICIENTS

The Rabinowitz–Floer Complex

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-,$$

$$d = \begin{pmatrix} d_{++} & 0 & 0 \\ d_{0+} & d_{00} & 0 \\ d_{-+} & d_{-0} & d_{--} \end{pmatrix} = \begin{pmatrix} d_{C^+ \oplus C^-} & 0 \\ \mathcal{CY}_{\mathcal{C},\mathcal{A}} & d_{--} \end{pmatrix} = \begin{pmatrix} d_{++} & 0 \\ \overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}} & d_{C^0 \oplus C^-} \end{pmatrix}$$

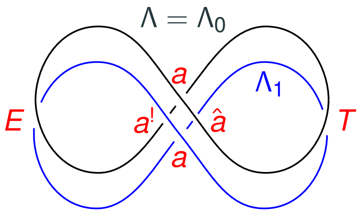
- The differential is a morphism of left $\mathcal{C}^e_{\mathcal{A}}$-modules (roughly speaking, it is defined by multiplying with elements from the “right”).
- The Rabinowitz–Floer complex has **two natural cone structures**:

$$\mathrm{Cone}\,(\mathcal{CY}_{\mathcal{C},\mathcal{A}}) = RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = \mathrm{Cone}\,(\overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}})$$

RABINOWITZ–FLOER COMPLEX WITH DGA COEFFICIENTS

The Rabinowitz–Floer Complex for the standard unknot

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-, \qquad d = \begin{pmatrix} d_{++} & 0 & 0 \\ d_{0+} & d_{00} & 0 \\ d_{-+} & d_{-0} & d_{--} \end{pmatrix}.$$



$$d_{++} = d_{--} = 0$$

$$d_{0+}(\hat{a}) = (1 \otimes_{\Bbbk} a - a \otimes_{\Bbbk} 1) \cdot E + (t \otimes_{\Bbbk} 1) \cdot T$$

$$d_{-+}(\hat{a}) = (a \otimes_{\Bbbk} a) \cdot a^!$$

$$d_{00}(T) = (1 \otimes_{\Bbbk} 1 - t^{-1} \otimes_{\Bbbk} t) \cdot E$$

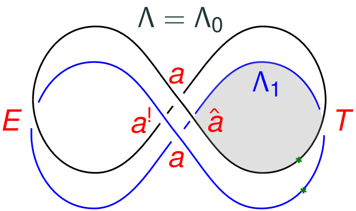
$$d_{-0}(E) = (1 \otimes_{\Bbbk} 1) \cdot a^!$$

$$d_{-0}(T) = (1 \otimes_{\Bbbk} a - t^{-1} a \otimes_{\Bbbk} t) \cdot a^!$$

RABINOWITZ–FLOER COMPLEX WITH DGA COEFFICIENTS

The Rabinowitz–Floer Complex for the standard unknot

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-, \qquad d = \begin{pmatrix} d_{++} & 0 & 0 \\ d_{0+} & d_{00} & 0 \\ d_{-+} & d_{-0} & d_{--} \end{pmatrix}.$$



$$d_{++} = d_{--} = 0$$

$$d_{0+}(\hat{a}) = (1 \otimes_{\mathbb{k}} a - a \otimes_{\mathbb{k}} 1) \cdot E + (t^{-1} \otimes_{\mathbb{k}} 1) \cdot T$$

$$d_{-+}(\hat{a}) = (a \otimes_{\mathbb{k}} a) \cdot a^!$$

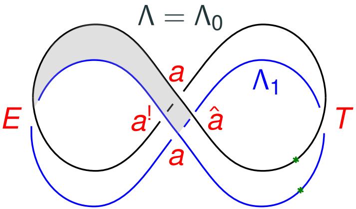
$$d_{00}(T) = (1 \otimes_{\mathbb{k}} 1 - t \otimes_{\mathbb{k}} t^{-1}) \cdot E$$

$$d_{-0}(E) = (1 \otimes_{\mathbb{k}} 1) \cdot a^!$$

$$d_{-0}(T) = (1 \otimes_{\mathbb{k}} a - ta \otimes_{\mathbb{k}} t^{-1}) \cdot a^!$$

THE RABINOWITZ–FLOER COMPLEX FOR THE STANDARD UNKNOT

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-, \qquad d = \begin{pmatrix} d_{++} & 0 & 0 \\ d_{0+} & d_{00} & 0 \\ d_{-+} & d_{-0} & d_{--} \end{pmatrix}.$$



$$d_{++} = d_{--} = 0$$

$$d_{0+}(\hat{a}) = (1 \otimes_{\Bbbk} a - a \otimes_{\Bbbk} 1) \cdot E + (t^{-1} \otimes_{\Bbbk} 1) \cdot T$$

$$d_{-+}(\hat{a}) = (a \otimes_{\Bbbk} a) \cdot a^!$$

$$d_{00}(T) = (1 \otimes_{\Bbbk} 1 - t \otimes_{\Bbbk} t^{-1}) \cdot E$$

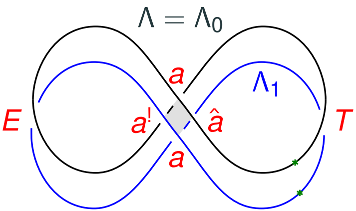
$$d_{-0}(E) = (1 \otimes_{\Bbbk} 1) \cdot a^!$$

$$d_{-0}(T) = (1 \otimes_{\Bbbk} a - ta \otimes_{\Bbbk} t^{-1}) \cdot a^!$$

RABINOWITZ–FLOER COMPLEX WITH DGA COEFFICIENTS

The Rabinowitz–Floer Complex for the standard unknot

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-, \qquad d = \begin{pmatrix} d_{++} & 0 & 0 \\ d_{0+} & d_{00} & 0 \\ d_{-+} & d_{-0} & d_{--} \end{pmatrix}.$$



$$d_{++} = d_{--} = 0$$

$$d_{0+}(\hat{a}) = (1 \otimes_{\Bbbk} a - a \otimes_{\Bbbk} 1) \cdot E + (t^{-1} \otimes_{\Bbbk} 1) \cdot T$$

$$d_{-+}(\hat{a}) = (a \otimes_{\Bbbk} a) \cdot a^!$$

$$d_{00}(T) = (1 \otimes_{\Bbbk} 1 - t \otimes_{\Bbbk} t^{-1}) \cdot E$$

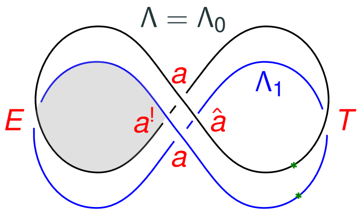
$$d_{-0}(E) = (1 \otimes_{\Bbbk} 1) \cdot a^!$$

$$d_{-0}(T) = (1 \otimes_{\Bbbk} a - ta \otimes_{\Bbbk} t^{-1}) \cdot a^!$$

RABINOWITZ–FLOER COMPLEX WITH DGA COEFFICIENTS

THE RABINOWITZ–FLOER COMPLEX FOR THE STANDARD UNKNOT

$$RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = C^+ \oplus C^0 \oplus C^-, \qquad d = \begin{pmatrix} d_{++} & 0 & 0 \\ d_{0+} & d_{00} & 0 \\ d_{-+} & d_{-0} & d_{--} \end{pmatrix}.$$



$$d_{++} = d_{--} = 0$$

$$d_{0+}(\hat{a}) = (1 \otimes_{\Bbbk} a - a \otimes_{\Bbbk} 1) \cdot E + (t^{-1} \otimes_{\Bbbk} 1) \cdot T$$

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$$d_{-0}(T) = (1 \otimes_{\Bbbk} a - ta \otimes_{\Bbbk} t^{-1}) \cdot a^!$$

$$
\begin{array}{ccccccc}
\cdots\to & \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}_* & \hookrightarrow & \mathcal{S}_* & \twoheadrightarrow & \overline{\mathcal{S}}_* & \overset{[-1]}{\cdots\to} \\
& \simeq \downarrow \mathcal{CY}_{\mathcal{A}} & & \simeq \downarrow \mathcal{CY}_{\mathcal{C},\mathcal{A}} & & \simeq \downarrow \overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}} & \\
\twoheadrightarrow & R\mathrm{Hom}^{*-n}_{\mathcal{C}^e_{\mathcal{A}}}(\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}, \mathcal{C}^e_{\mathcal{A}}) & \overset{[-1]}{\cdots\to} & R\mathrm{Hom}^{*-n-1}_{\mathcal{C}^e_{\mathcal{A}}}(\overline{\mathcal{S}}, \mathcal{C}^e_{\mathcal{A}}) & \hookrightarrow & R\mathrm{Hom}^{*-n-1}_{\mathcal{C}^e_{\mathcal{A}}}(\mathcal{S}, \mathcal{C}^e_{\mathcal{A}}) & \to
\end{array}
$$

- The leftmost vertical arrow is the absolute CY-structure on $C_*\Omega(\Lambda)$.
- Computing the Rabinowitz Floer complexes we recover the sought cones of the resolutions:

$$\mathrm{Cone}\,(\mathcal{CY}_{\mathcal{C},\mathcal{A}}) = RFC_*(\Lambda_0, \Lambda_1; \mathcal{C}^e_{\mathcal{A}}) = \mathrm{Cone}\,(\overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}})$$

THE PROOF

- The Rabinowitz Floer complex is **acyclic** in boundaries of subcritical Weinstein manifolds.
- In the standard Darboux ball any Legendrian is even horizontally displaceable. Here the acyclicity follows by invariance under Legendrian isotopy.
- Acyclicity implies that the vertical maps are quasi-isomorphsims.

$$
\begin{array}{ccccc}
\cdots\to \mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}_* & \hookrightarrow & \mathcal{S}_* & \twoheadrightarrow & \overline{\mathcal{S}}_* \overset{[-1]}{\cdots\to} \\
\simeq \downarrow \mathcal{CY}_{\mathcal{A}} & & \simeq \downarrow \mathcal{CY}_{\mathcal{C},\mathcal{A}} & & \simeq \downarrow \overline{\mathcal{CY}}_{\mathcal{C},\mathcal{A}} \\
\twoheadrightarrow R\mathrm{Hom}^{*-n}_{\mathcal{C}^e_{\mathcal{A}}}(\mathcal{C}^e_{\mathcal{A}} \otimes_{\mathcal{A}^e} \mathcal{R}, \mathcal{C}^e_{\mathcal{A}}) & \overset{[-1]}{\cdots\to} & R\mathrm{Hom}^{*-n-1}_{\mathcal{C}^e_{\mathcal{A}}}(\overline{\mathcal{S}}, \mathcal{C}^e_{\mathcal{A}}) & \hookrightarrow & R\mathrm{Hom}^{*-n-1}_{\mathcal{C}^e_{\mathcal{A}}}(\mathcal{S}, \mathcal{C}^e_{\mathcal{A}}) \longrightarrow
\end{array}
$$

Taking the tensor product $(\mathbb{K}^\vee_{\varepsilon_0} \otimes_{\Bbbk} \mathbb{K}_{\varepsilon_1}) \otimes_{\mathcal{C}^e_{\mathcal{A}}} \cdot$ produces:

$$
\begin{array}{ccccc}
C_{*-1}(\Lambda; \nabla_0 \otimes_{\Bbbk} \nabla_1^{-1}) & \hookrightarrow & \mathrm{Hom}^*_{Aug_+}(\varepsilon_1, \varepsilon_0)^\vee & \twoheadrightarrow & \mathrm{Hom}^*_{Aug_-}(\varepsilon_1, \varepsilon_0)^\vee \\
\downarrow \mathrm{PD} & & \downarrow q.is & & \downarrow q.is \\
\cdots \twoheadrightarrow C^{n-*+1}(\Lambda; \nabla_0 \otimes_{\Bbbk} \nabla_1^{-1}) & & \mathrm{Hom}^{n-*+1}_{Aug_-}(\varepsilon_0, \varepsilon_1) & \hookrightarrow & \mathrm{Hom}^{n-*+1}_{Aug_+}(\varepsilon_0, \varepsilon_1) \longrightarrow \cdots
\end{array}
$$

Part V: APPLICATIONS

FUNDAMENTAL CLASS

- Canonical classes in $\mathcal{R}_*$ and $(\mathcal{R}_*)^!$ (typically appear after tensoring with a bimodule) give rise to canonical classes in $\mathcal{S}_*$ and $\mathcal{S}^!$, respectively.
- A non-trivial fundamental class in $\mathcal{S}_*$ (after tensoring with a bimodule) implies non-surjectivity onto $\overline{\mathcal{S}}_*$. This can be used to show that the augmentation variety is of positive codimension; c.f. the even stronger result for Legendrian knots in $\mathbb{R}^3$ by Henry–Rutherford [HR15] and Leverson [Lev16].
- Under additional assumptions, exact Lagrangian cobordisms induce "Calabi–Yau morphisms" that respect the fundamental class in $\mathcal{S}^!$. This can be used to generalise Pan's result [Pan17] about cobordism maps being injective on the augmentation variety. Also see [CSLL$^+$22].

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