14/04/2023

Quivers, flow trees, and log curves. [Pierrick Bousseau

Symplectic Zoominar   Joint work with H. Arguz 2302.02068

① Donaldson-Thomas invariants.

Geometry   X Calabi-Yau 3-fold

Holomorphic vector bundles

Stable

↕ Mirror symmetry

Lagrangian submanifolds

Special

TODAY : ALGERAIC TOY MODEL

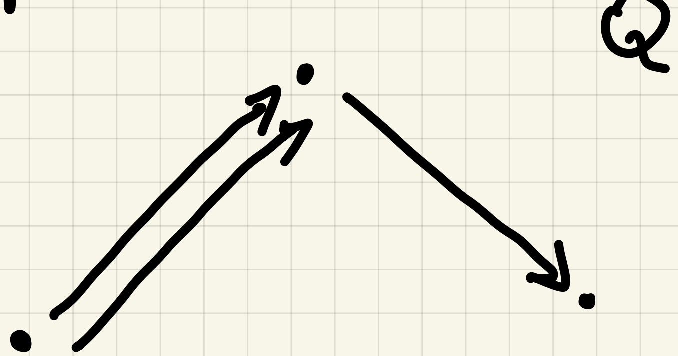
$\#$ Stable $\underbrace{\text{quiver}}_{\text{QUIVER}}$ representations.

② COUNTING HOLOMORPHIC CURVES

~ GROMOV-WITTEN

IN TORIC VARIETIES

① Quiver



Quiver representation:

$$V = \left(V_i \ , \quad f_\alpha : V_i \longrightarrow V_j \right)$$

f.d $\mathbb{C}$ vector space for each vertex i

$\forall \alpha : i \to j$

Abelian category

$$\dim V = (\dim V_i) \in N = \bigoplus_i \mathbb{Z} e_i$$

$$\overset{d \text{ vertices}}{\mathbb{Z}^d} \cong N$$

$\gamma \in N \quad \gamma = (\gamma_i)$

$$\left\{ \begin{matrix} \text{Representations} \\ \text{of } \dim \gamma \end{matrix} \right\} / \text{Isom} = \left(\bigoplus_{\alpha : i \to j} \operatorname{Hom}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j}) \right) \Big/ \prod_i GL(\gamma_i, \mathbb{C})$$

King's stability

$$M = \operatorname{Hom}(N, \mathbb{Z})$$

$$M_{\mathbb{R}} = M \otimes \mathbb{R} = \operatorname{Hom}(N, \mathbb{R})$$

$\gamma \in N$

$$\theta \in \gamma^{\perp} = \{ \theta \mid \theta(\gamma) = 0 \} \subset M_{\mathbb{R}}$$

"Stability parameter".

$V \quad \dim V = \gamma$

V θ-semistable if $\theta(\dim V') \leq 0$ $\forall\, V' \subset V$.

Ex: V $\qquad$ V_i', f_α' $\qquad$ V_i, f_α

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "1"] & \mathbb{C} \\
\bullet \arrow[r] & \bullet \\
0 \arrow[u, "1"] & \mathbb{C} \arrow[u, "1"] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
V_i' \arrow[r, phantom, "\subset"] \arrow[d, "f_\alpha'"'] & V_i \arrow[d, "f_\alpha"] \\
V_j' \arrow[r, phantom, "\subset"] & V_j
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "1"] & \mathbb{C} \\
\mathbb{C} \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

$$M_\gamma^\theta = \left\{ \begin{array}{c} \theta\text{-semistable} \\ \text{representations} \\ \text{of dim } \gamma \end{array} \right\}$$

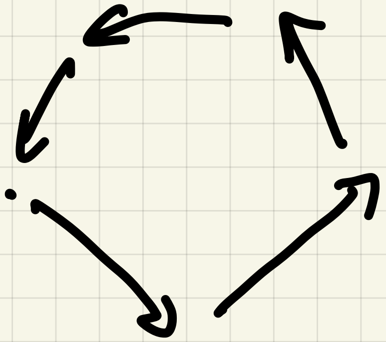
$\uparrow$

Quasiprojective variety $/\mathbb{C}$

GIT

Projective if Q is acyclic

If Q has oriented cycles

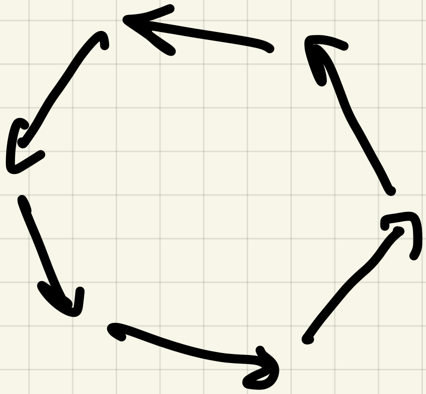


Pick a potential W

formal linear combination

of oriented cycles in Q.

$$\mathrm{Tr}\, W : M_\gamma^\theta \longrightarrow \mathbb{C}$$



DT invariants

Ω_γ^θ = "Virtual counts of critical points of $\mathrm{Tr}\, W$ on M_γ^θ".

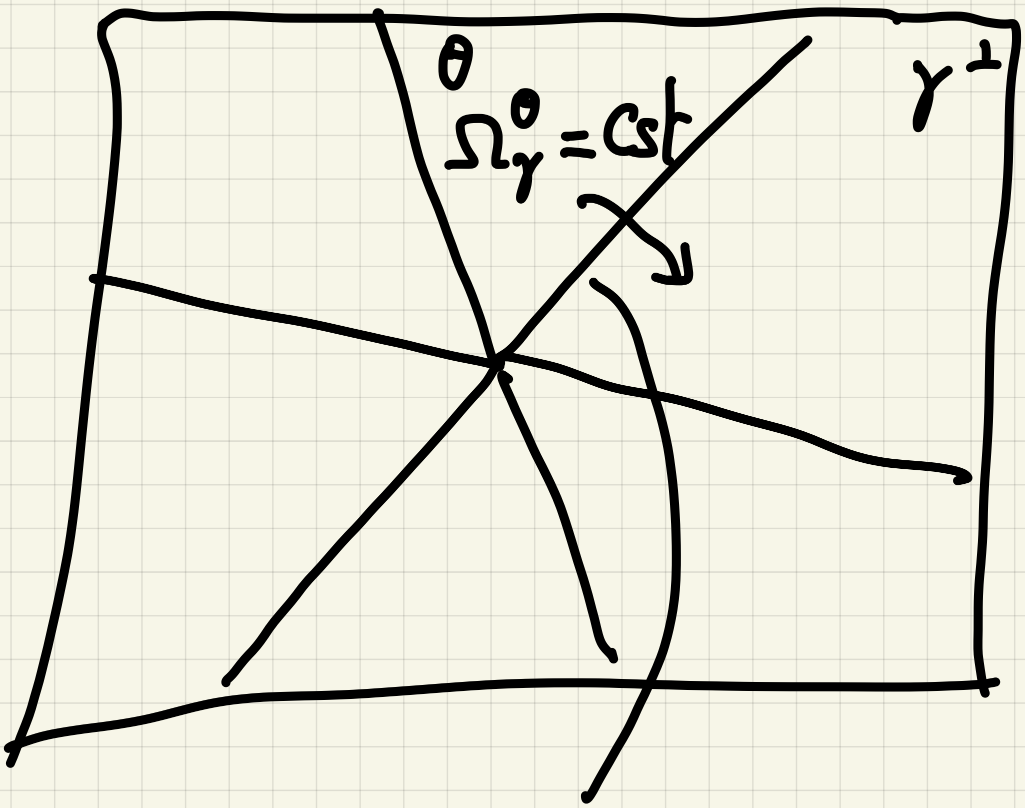
$$\Omega_\gamma^\theta = e\left(M_\gamma^\theta, \phi_{\mathrm{Tr} W}\left(IC_{M_\gamma^\theta}\right)\right) \in \mathbb{Z}$$

Q W

γ θ

$$\bar{\Omega}_\gamma^\theta = \sum_{\substack{\gamma = k\gamma' \\ k \in \mathbb{Z}_{\geq 1}}} \frac{(-1)^{k-1}}{k^2} \Omega_{\gamma'}^\theta \in \mathbb{Q}$$

$\Omega_\gamma^\theta \leftarrow$ Stability $\theta \in \gamma^\perp \subset M_{\mathbb{R}} \simeq \mathbb{R}^d$



WALL-CROSSING
STORY

KONTSEVICH-SOIBELMAN
WALL-CROSSING FORMULA

Express general DT invariants Ω_γ^θ in terms of attractor DT invariants

$N = \bigoplus_i \mathbb{Z} e_i \qquad \gamma^{\perp} \subset M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R})$

$\omega : N \times N \to \mathbb{Z} \qquad \omega(e_i, e_j) = a_{ij} - a_{ji}$

$a_{ij} = \# \, i \to j$

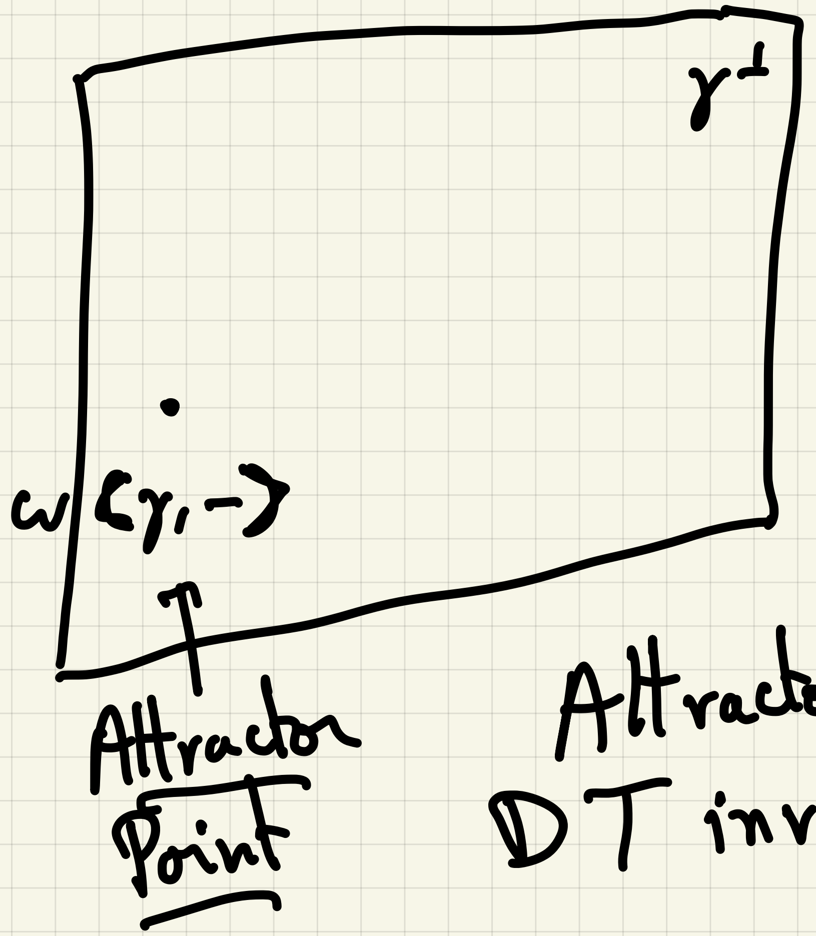
$$\left[\begin{array}{l} X \text{ CY 3-fold} \\ [C] \in H_3(X) \\ H_3(X) \times H_3(X) \to \mathbb{Z} \end{array} \right]$$

$\gamma \in N$

$\omega(\gamma, -) \in M_{\mathbb{R}}$

$\in \gamma^{\perp} \subset$

because $\omega(\gamma, \gamma) = 0$



$$\Omega^{*}_{\gamma} = \Omega_{\gamma}^{\omega(\gamma, -)}$$

Attractor
DT invariant

Geometric: X Calabi-Yau 3-folds

$$\underset{\underset{[L]}{\shortparallel}}{\gamma} \in H_3(X) \simeq N$$

$$\{\text{Complex structures on } X\}$$

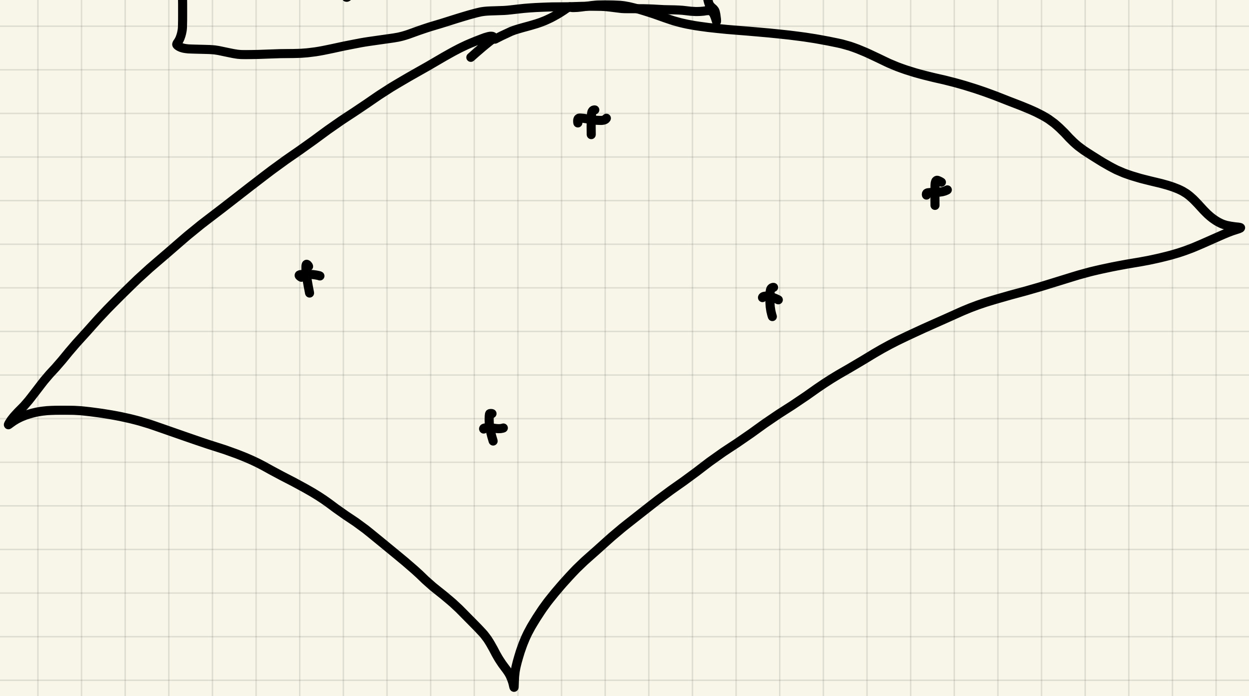
$$\uparrow \; \dim_{\mathbb{C}} = h^{1,2}(X)$$

Attractor condition: $\gamma \in H^3(X)$

$$\shortparallel$$

$$H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

$$\boxed{\gamma|_{H^{2,1}} = 0}$$



Q $\quad \Omega^*_\gamma$

Often simple:

- If Q acyclic, $\Omega^*_\gamma = \begin{cases} 0 & \text{if } \gamma \neq e_i \\ 1 & \text{if } \gamma = e_i \end{cases}$

- Conj (Mozgovoy-Pioline)

X toric Calabi-Yau 3-fold

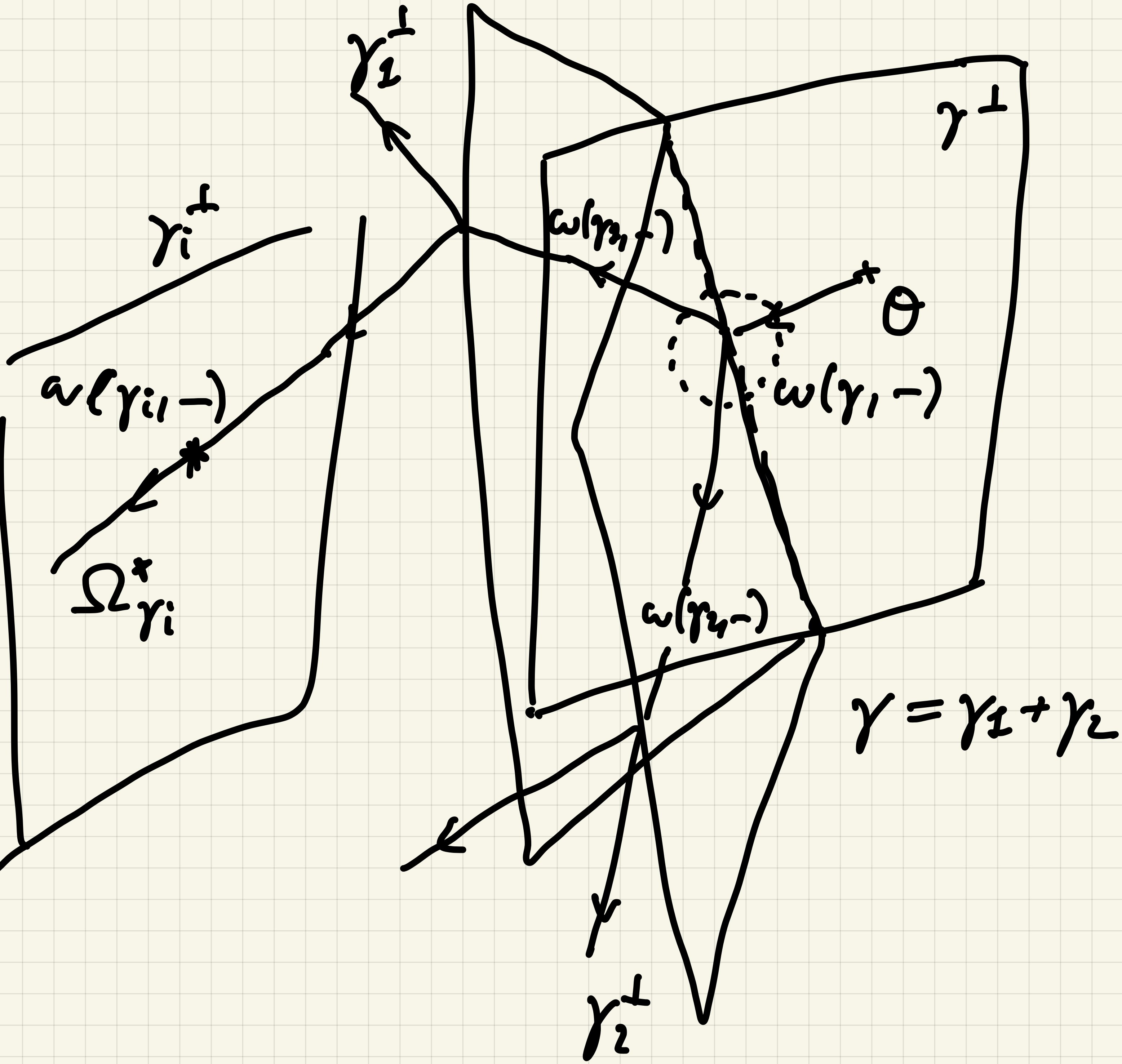
(Q,W) $\qquad D^b\,\mathrm{Coh}(X)$

$$\Omega^*_\gamma = \begin{cases} 0 & \text{unless } \gamma = e_i \\ & \text{or } \gamma \in \mathrm{Ker}\langle , \rangle \end{cases}$$

True for $X = K_{\mathbb{P}^2}$ [B-Descombes-Le Floch-Pioline]

WALL-CROSSING

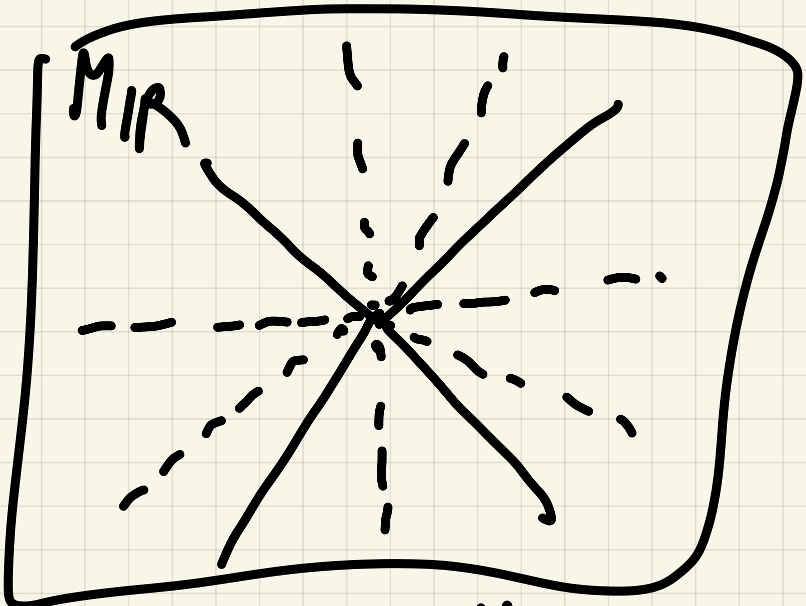
$$\hookrightarrow \bar{\Omega}^{\boxed{\theta}}_\gamma = \sum_{\gamma = \gamma_1 + \dots + \gamma_r} \sum_{T \text{ trees}} \underbrace{\boxed{F^\theta_{r,T}(\gamma_1, \dots, \gamma_r)}}_{\in \mathbb{Z}} \prod_{i=1}^{r} \bar{\Omega}^*_{\gamma_i}$$


$\gamma_1^\perp$
$\gamma^\perp$
$\gamma_i^\perp$
$\omega(\gamma_1,-)$
θ
$\omega(\gamma_i,-)$
$\omega(\gamma_1,-)$
$\Omega^*_{\gamma_i}$
$\omega(\gamma_2,-)$
$\gamma = \gamma_1 + \gamma_2$
$\gamma_2^\perp$

Toric variety $\dim_{\mathbb{C}} = d$ $d = \#$ vertices of Q

FAN in $M_{\mathbb{R}} = \mathrm{Hom}(N, \mathbb{R}) \simeq \mathbb{R}^d$ $\dim_{\mathbb{C}} = h^{31}$

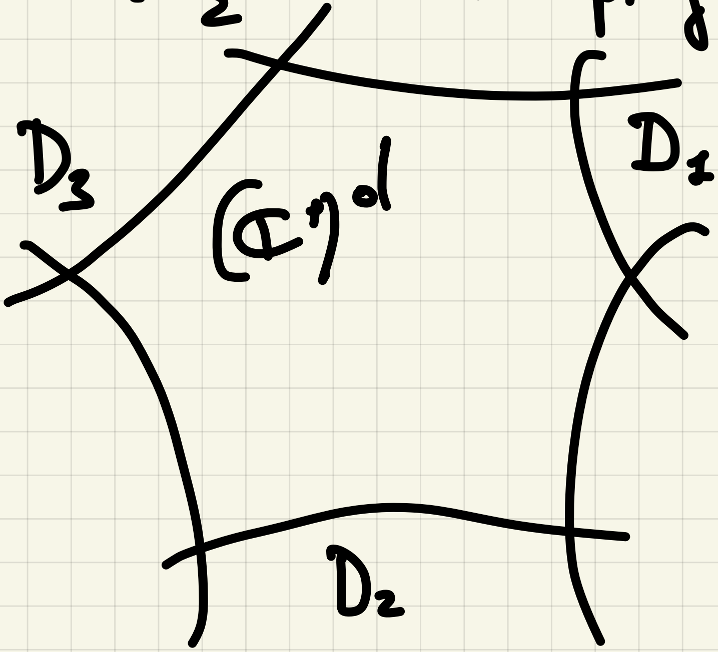
Σ containing rays $\mathbb{R}_{\geq 0}\, \omega(\gamma_i, -) \leftrightarrow D_i$

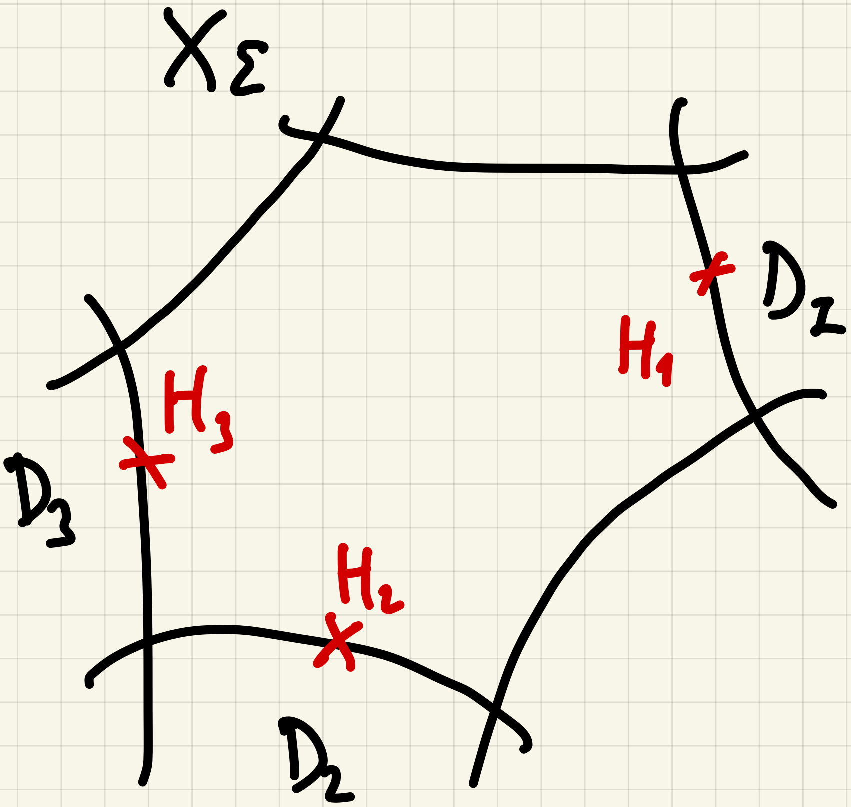


Fix γ

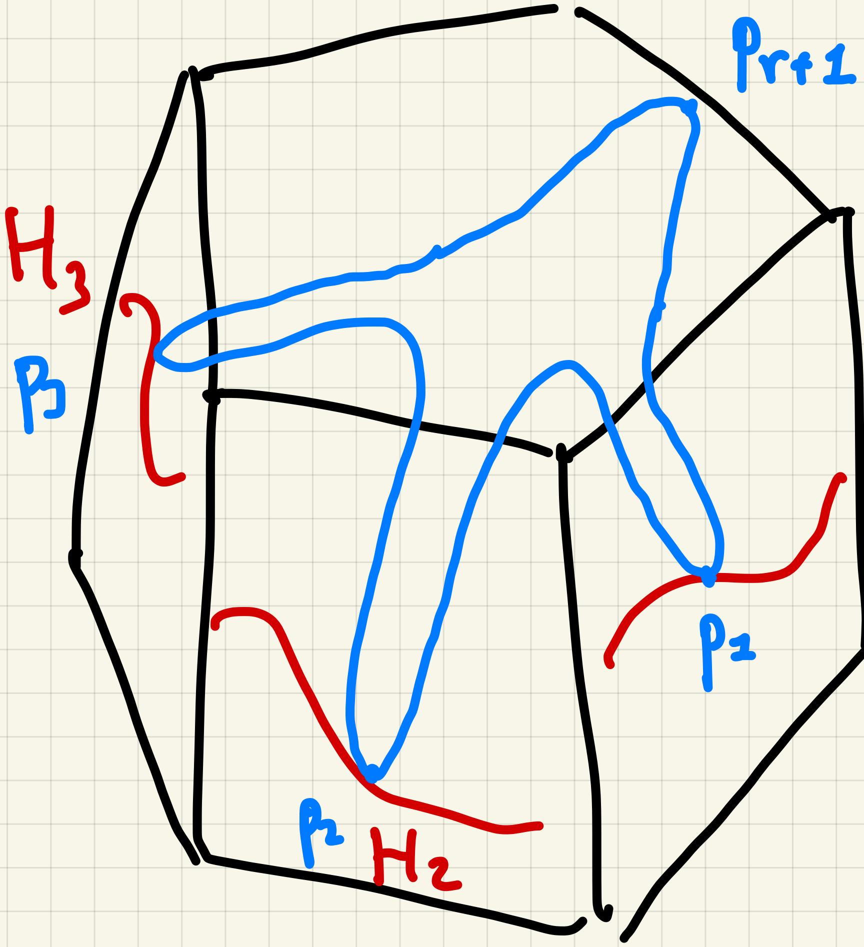
$\gamma = \gamma_1 + \dots + \gamma_r$

X_Σ smooth proj d-dim toric variety





$$D_i$$
$$\cup \leftarrow \text{Hypersurfa}$$
$$\text{in } D_i$$
$$\boxed{H_i}$$
$$\|$$
$$\left\{ \left(z^{\frac{\gamma_i}{|\gamma_i|}} = c_i \right) \Big|_{D_i} \right\}$$



$$\mathcal{M} = \Big\{ f :$$

$$\big(\underset{\overset{\uparrow}{g=0}}{C}, p_1, \ldots, p_r, \boxed{p_{r+1}} \big) \longrightarrow X_\Sigma,$$

$$f^{-1}(\partial X_\Sigma) = \{ p_1, \ldots, p_r, p_{r+1} \}$$

$$f(p_i) \in H_i \quad 1 \leq i \leq r$$

At p_i, contact order of C with $D_i = |\omega(\gamma_i, -)|$

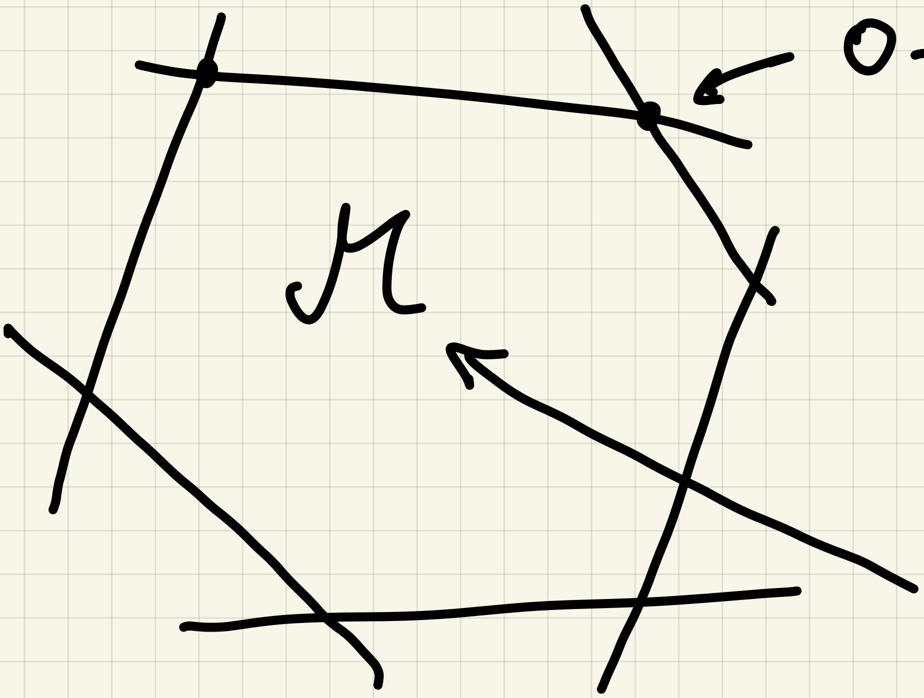
exp dim $\mathcal{M} = d-2$

$\mathcal{M}$ is not compact.

$\overline{\mathcal{M}}$ obtained as moduli space of stable log maps (Abramovich-Chen Gross-Siebert)

Log Gromov-Witten theory

Picture of $\overline{\mathcal{M}}$? (Virtually)

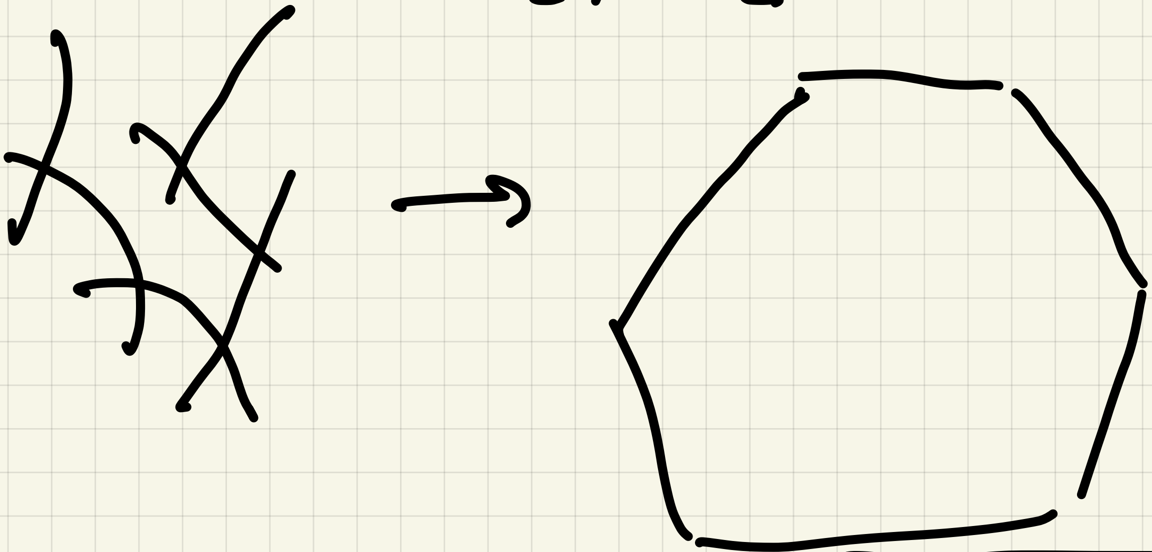


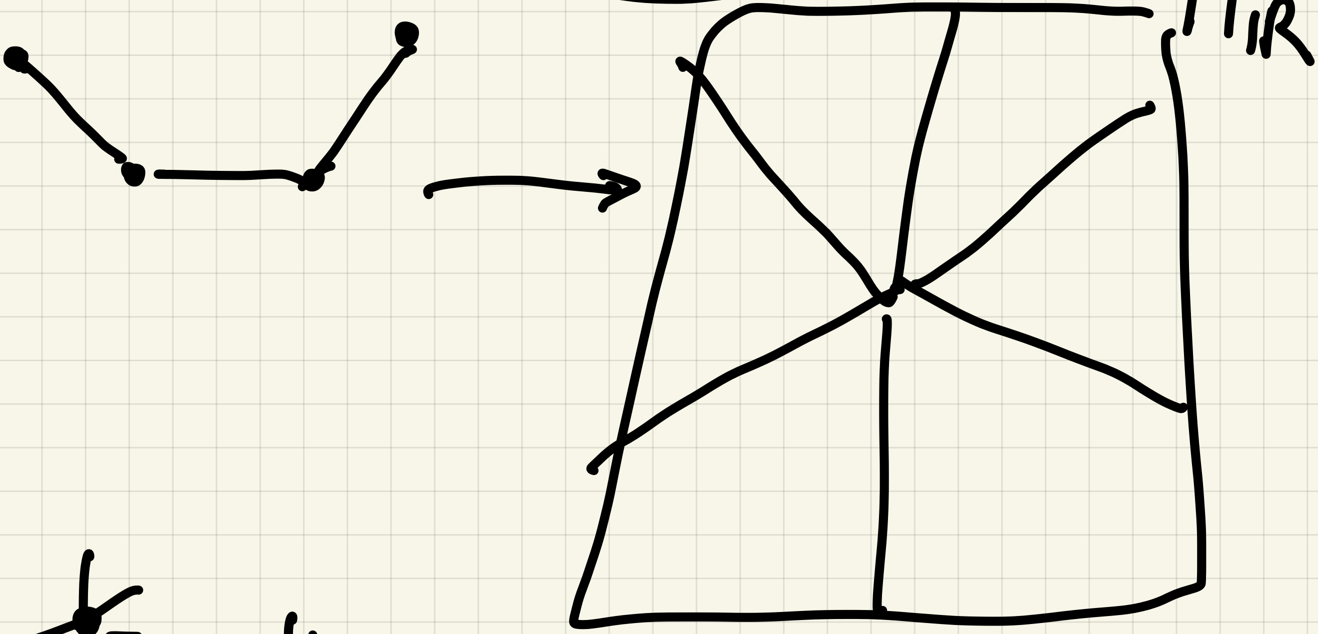
0-dim strata where $(d-2)$-divisors intersect Transversally

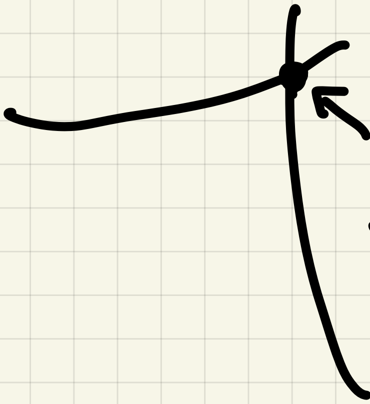
dim $d-2$

Strata of $\mathcal{M}$ $\rightsquigarrow$ Family of tropical cures in $M_{\mathbb{R}}$

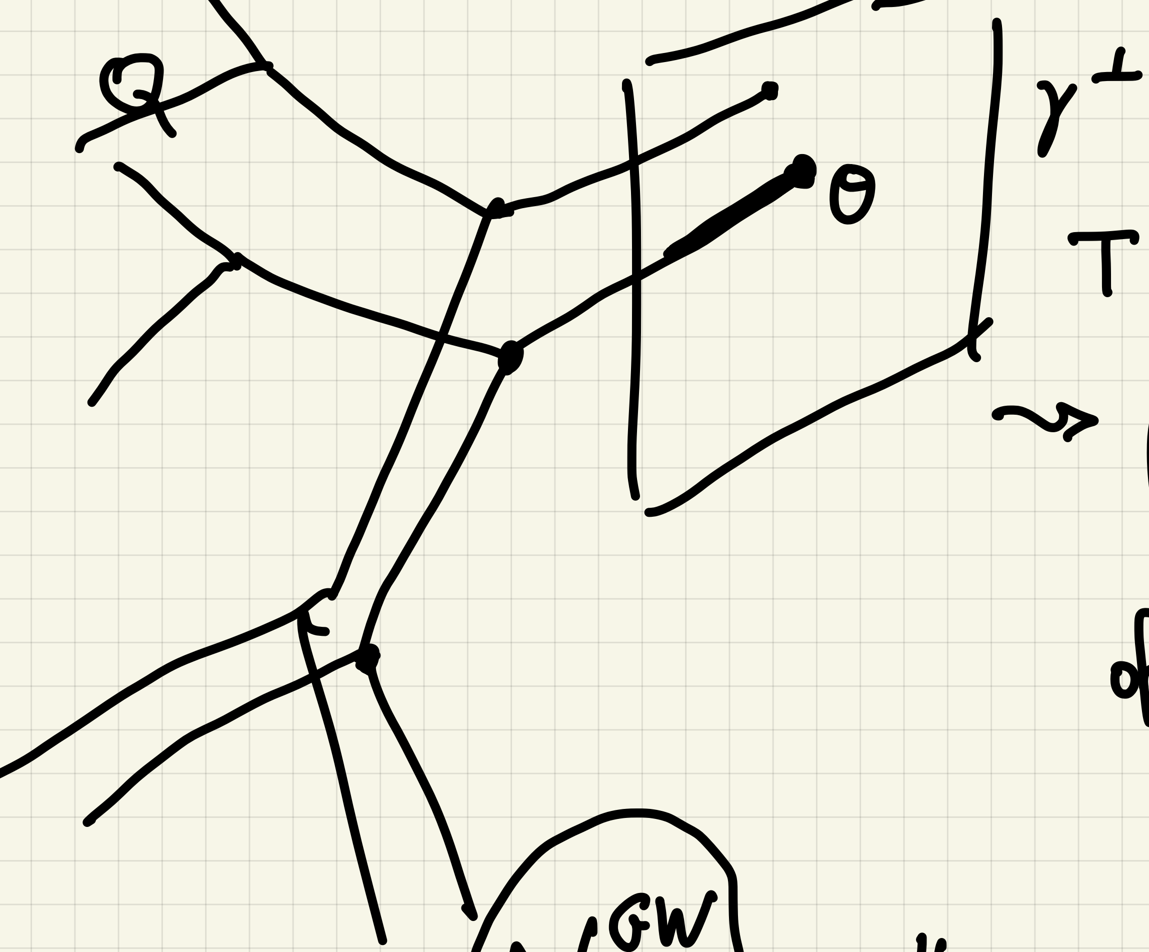
$$C \longrightarrow X_\Sigma, \partial X_\Sigma$$







0-dim
Strata
$\updownarrow$ (d-2)-dim
families of
Tropical curves



T tree

$\rightsquigarrow \rho_T$ (d-2)-family of tropical curves.

[$N^{GW}_{\rho_T}$ = # 0-dim strata in moduli space $\mathcal{M}$ with tropicalization ρ_T]

[Thm: [Argüz-B]

$$\boxed{F_T = N^{GW}_{\rho_T}}$$
]

$$\begin{tikzcd} F_T & N^{GW}_{\rho_T} \arrow[d] \\ e(\mathcal{M}_\gamma) \arrow[u] & ??? \end{tikzcd}$$

$H^*(\mathcal{M}_\gamma)$ ...