Gromov width of disk cotangent bundles of spheres of revolution

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Symplectic Zoominar

Joint work with Vinicius Ramos and Alejandro Vicente

Gromov's Nonsqueezing

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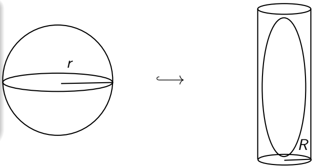

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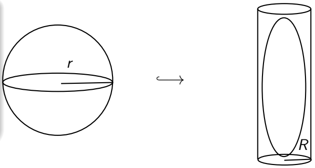

Symplectic embeddings $\neq$ Volume preserving embeddings

Symplectic capacities - Definition

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- $c(M, \alpha\omega) = |\alpha| c(M, \omega)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$,
- $c(B^{2n}(r), \omega_0) > 0$ and $c(Z^{2n}(r), \omega_0) < \infty$.

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- ECH capacities c_k^{ECH} (Hutchings) - only in dimension 4.

Examples of ECH capacities

- Ellipsoids (Hutchings):

$$E(a,b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \pi\left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\right) < 1 \right\}.$$

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  It agrees with the "even multiples" that appears in the sequence for the ball $B(\ell)$.

Toric domains

A subset $\Omega \subset (\mathbb{R}_{\geq 0})^2$ gives rise to a domain:

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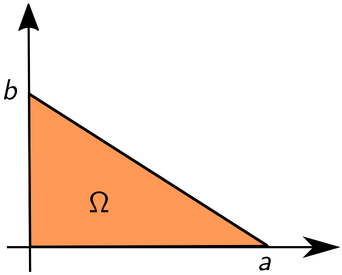

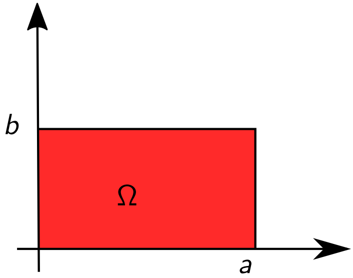

Integrable systems in spheres of revolution

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Proof Idea: Action angle coordinates from Arnold–Liouville Theorem for the perturbed system:

$$H_\varepsilon(q,p) = \|p\|^2 + U_\varepsilon(q) \quad J(q,p) = p(\partial_\theta),$$

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Nested domains $H_\varepsilon^{-1}([0,1)) \cong X_{\Omega_\varepsilon}$ converging to $D^*(S \setminus \{P_N\}) \cong \operatorname{int} X_{\Omega_S}$ when $\varepsilon \to 0$. □

Ellipsoids of revolution

For $a, b, c > 0$, let $\mathcal{E}(a, b, c) \subset \mathbb{R}^3$ be the ellipsoid defined by the equation:

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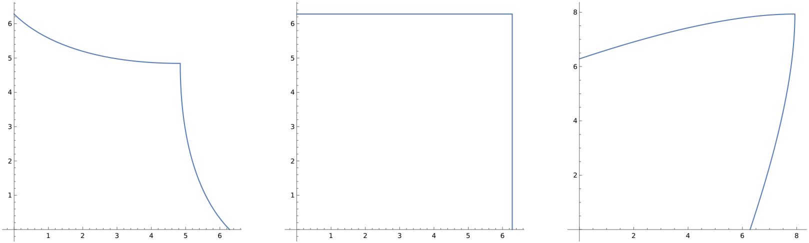

Figure: The region $\Omega_{\mathcal{E}(1,1,c)}$ for $c = 0,5$; $c = 1$; $c = 1,5$, respectively.

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The Gromov width of $D^*\mathcal{E}(1,1,c)$ is given by

$$c_{Gr}(D^*\mathcal{E}(1,1,c),\omega_{can}) = \begin{cases} \alpha(c), & \text{for } 0 < c < 1/2, \\ 2\pi, & \text{for } 1/2 \le c \le 1, \\ \beta(c), & \text{for } 1 < c < c_0, \\ 4\pi, & \text{for } c \ge c_0. \end{cases}$$

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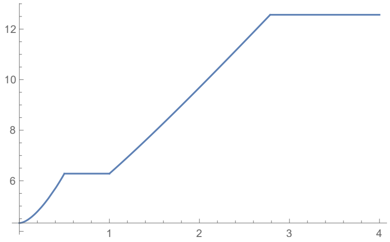

Figure: Graph of function $c \mapsto c_{Gr}(D^*\mathcal{E}(1,1,c),\omega_{can})$.

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Let S be a Zoll sphere of revolution and ℓ be the length of any simple closed geodesic. Then

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On other hand, if $(B(a), \omega_0) \hookrightarrow (D^*S, \omega_{can})$, we have

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It yields $c_{Gr}(D^*S, \omega_{can}) \leq \ell$. $\square$

Thank you!