Maslov index formula in Heegaard Floer homology

Roman Krutowski

University of California, Los Angeles

Symplectic Zoominar
21 April, 2023

Outline

1 Heegaard Floer Homology

Outline

1. Heegaard Floer Homology

2. Maslov index formula

Outline

1. Heegaard Floer Homology

2. Maslov index formula

3. Combinatorial proof

Heegaard diagram (unpointed)

Setup

- Let Σ be a surface of genus g with a metric

Heegaard diagram (unpointed)

Setup

- Let Σ be a surface of genus g with a metric
- Let $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_k\}$ be two sets of non-intersecting closed simple curves in Σ (here $k \geqslant g$). Assume that these curves intersect at 90° angles.

Heegaard diagram (unpointed)

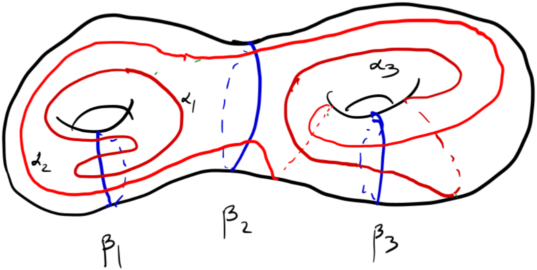
Setup

- Let Σ be a surface of genus g with a metric
- Let $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ and $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_k\}$ be two sets of non-intersecting closed simple curves in Σ (here $k \geqslant g$). Assume that these curves intersect at 90° angles.
- In our notation $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an (*unpointed*) Heegaard diagram.

Remark

In literature, by Heegaard diagram people call the above data together with a collection of $k - g + 1$ points in different connected regions of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$.

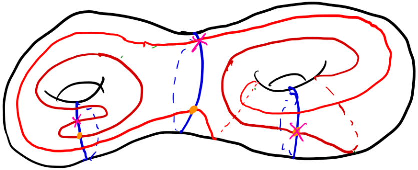
Heegaard diagram: example



Heegaard Floer chain complex

Generators

Each collection of points $\boldsymbol{x} = \{x_1, \ldots, x_k\}$ where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$, $\sigma \in S_k$ serves as a generator.
It can be regarded as a point $\boldsymbol{x} \in T_{\boldsymbol{\alpha}} \cap T_{\boldsymbol{\beta}} \subset Sym^k(\Sigma)$.



Differential

- Consider a space $W = \mathbb{R} \times [0,1] \times \Sigma$ with a *nice* almost complex structure J. Let $C_\alpha = \mathbb{R} \times \{1\} \times \boldsymbol{\alpha}$ and $C_\beta = \mathbb{R} \times \{0\} \times \boldsymbol{\beta}$.

Differential

- Consider a space $W = \mathbb{R} \times [0,1] \times \Sigma$ with a *nice* almost complex structure J. Let $C_\alpha = \mathbb{R} \times \{1\} \times \boldsymbol{\alpha}$ and $C_\beta = \mathbb{R} \times \{0\} \times \boldsymbol{\beta}$.
- Let $\mathcal{M}$ be a moduli space of Riemann surfaces $(S, \partial S)$ with k negative **boundary punctures** $\boldsymbol{p} = \{p_1, \ldots, p_k\}$, k positive **boundary punctures** $\boldsymbol{q} = \{q_1, \ldots, q_k\}$, and such that S is compact away from punctures.

Differential

- Consider a space $W = \mathbb{R} \times [0,1] \times \Sigma$ with a *nice* almost complex structure J. Let $C_\alpha = \mathbb{R} \times \{1\} \times \boldsymbol{\alpha}$ and $C_\beta = \mathbb{R} \times \{0\} \times \boldsymbol{\beta}$.
- Let $\mathcal{M}$ be a moduli space of Riemann surfaces $(S, \partial S)$ with k negative **boundary punctures** $\boldsymbol{p} = \{p_1, \ldots, p_k\}$, k positive **boundary punctures** $\boldsymbol{q} = \{q_1, \ldots, q_k\}$, and such that S is compact away from punctures.
- For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ denote by $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ the set of homology classes of maps $(S, \partial S) \to (W, C_\alpha \cup C_\beta)$ converging to $\boldsymbol{x}$ (to $\boldsymbol{y}$) near the negative punctures (positive punctures) where $(S, \partial S) \in \mathcal{M}$.

Differential

- Consider a space $W = \mathbb{R} \times [0,1] \times \Sigma$ with a *nice* almost complex structure J. Let $C_\alpha = \mathbb{R} \times \{1\} \times \boldsymbol{\alpha}$ and $C_\beta = \mathbb{R} \times \{0\} \times \boldsymbol{\beta}$.
- Let $\mathcal{M}$ be a moduli space of Riemann surfaces $(S, \partial S)$ with k negative **boundary punctures** $\boldsymbol{p} = \{p_1, \ldots, p_k\}$, k positive **boundary punctures** $\boldsymbol{q} = \{q_1, \ldots, q_k\}$, and such that S is compact away from punctures.
- For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ denote by $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ the set of homology classes of maps $(S, \partial S) \to (W, C_\alpha \cup C_\beta)$ converging to $\boldsymbol{x}$ (to $\boldsymbol{y}$) near the negative punctures (positive punctures) where $(S, \partial S) \in \mathcal{M}$.
- For $\varphi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ consider a moduli space $\mathcal{M}^\varphi$ of J-holomorphic curves connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ of class φ.

Differential

- Consider a space $W = \mathbb{R} \times [0,1] \times \Sigma$ with a *nice* almost complex structure J. Let $C_\alpha = \mathbb{R} \times \{1\} \times \boldsymbol{\alpha}$ and $C_\beta = \mathbb{R} \times \{0\} \times \boldsymbol{\beta}$.
- Let $\mathcal{M}$ be a moduli space of Riemann surfaces $(S, \partial S)$ with k negative **boundary punctures** $\boldsymbol{p} = \{p_1, \ldots, p_k\}$, k positive **boundary punctures** $\boldsymbol{q} = \{q_1, \ldots, q_k\}$, and such that S is compact away from punctures.
- For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ denote by $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ the set of homology classes of maps $(S, \partial S) \to (W, C_\alpha \cup C_\beta)$ converging to $\boldsymbol{x}$ (to $\boldsymbol{y}$) near the negative punctures (positive punctures) where $(S, \partial S) \in \mathcal{M}$.
- For $\varphi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ consider a moduli space $\mathcal{M}^\varphi$ of J-holomorphic curves connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ of class φ.
- Differential $\partial(\boldsymbol{x})$ in CF (and its versions) is given by the count of points in $\mathcal{M}^\varphi/\mathbb{R}$ for those $\boldsymbol{y}$ and φ satisfying $dim(\mathcal{M}^\varphi) = 1$ (sometime other dimensions are considered as well).

Differential

- Consider a space $W = \mathbb{R} \times [0,1] \times \Sigma$ with a *nice* almost complex structure J. Let $C_\alpha = \mathbb{R} \times \{1\} \times \boldsymbol{\alpha}$ and $C_\beta = \mathbb{R} \times \{0\} \times \boldsymbol{\beta}$.
- Let $\mathcal{M}$ be a moduli space of Riemann surfaces $(S, \partial S)$ with k negative **boundary punctures** $\boldsymbol{p} = \{p_1, \ldots, p_k\}$, k positive **boundary punctures** $\boldsymbol{q} = \{q_1, \ldots, q_k\}$, and such that S is compact away from punctures.
- For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ denote by $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ the set of homology classes of maps $(S, \partial S) \to (W, C_\alpha \cup C_\beta)$ converging to $\boldsymbol{x}$ (to $\boldsymbol{y}$) near the negative punctures (positive punctures) where $(S, \partial S) \in \mathcal{M}$.
- For $\varphi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ consider a moduli space $\mathcal{M}^\varphi$ of J-holomorphic curves connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ of class φ.
- Differential $\partial(\boldsymbol{x})$ in CF (and its versions) is given by the count of points in $\mathcal{M}^\varphi/\mathbb{R}$ for those $\boldsymbol{y}$ and φ satisfying $dim(\mathcal{M}^\varphi) = 1$ (sometime other dimensions are considered as well).
- Due to Ozsváth-Szabó $dim(\mathcal{M}^\varphi)$ only depends on φ. It is called *Maslov index* of φ and denoted $\mu(\varphi)$.

Table of Contents

1. Heegaard Floer Homology
2. Maslov index formula
3. Combinatorial proof

Maslov index formula

Motivation

Robert Lipshitz found a way to compute $\mu(\varphi)$ for $\varphi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ in terms of combinatorial data of a Heegaard diagram. This formula is now used everywhere.

- In the interior of each region R_i of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ pick a point ζ_i

Maslov index formula

Motivation

Robert Lipshitz found a way to compute $\mu(\varphi)$ for $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$ in terms of combinatorial data of a Heegaard diagram. This formula is now used everywhere.

- In the interior of each region R_i of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ pick a point ζ_i
- We put $n_{R_i}(\varphi)$ to be the *intersection number* between φ and $\mathbb{R} \times [0,1] \times \{\zeta_i\}$

Maslov index formula

Motivation

Robert Lipshitz found a way to compute $\mu(\varphi)$ for $\varphi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ in terms of combinatorial data of a Heegaard diagram. This formula is now used everywhere.

- In the interior of each region R_i of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ pick a point ζ_i
- We put $n_{R_i}(\varphi)$ to be the *intersection number* between φ and $\mathbb{R} \times [0,1] \times \{\zeta_i\}$
- For $p \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ define $n_p(\varphi)$ as an average of $n_R(\varphi)$ for those 4 regions to which p belongs

Motivation

Robert Lipshitz found a way to compute $\mu(\varphi)$ for $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$ in terms of combinatorial data of a Heegaard diagram. This formula is now used everywhere.

- In the interior of each region R_i of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ pick a point ζ_i
- We put $n_{R_i}(\varphi)$ to be the *intersection number* between φ and $\mathbb{R} \times [0, 1] \times \{\zeta_i\}$
- For $p \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ define $n_p(\varphi)$ as an average of $n_R(\varphi)$ for those 4 regions to which p belongs
- A *shadow* of φ is a 2-chain $D(\varphi) = \sum_R n_R(\varphi)R$.

Motivation

Robert Lipshitz found a way to compute $\mu(\varphi)$ for $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$ in terms of combinatorial data of a Heegaard diagram. This formula is now used everywhere.

- In the interior of each region R_i of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ pick a point ζ_i
- We put $n_{R_i}(\varphi)$ to be the *intersection number* between φ and $\mathbb{R} \times [0,1] \times \{\zeta_i\}$
- For $p \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ define $n_p(\varphi)$ as an average of $n_R(\varphi)$ for those 4 regions to which p belongs
- A *shadow* of φ is a 2-chain $D(\varphi) = \sum_R n_R(\varphi) R$.
- For a region R the Euler measure $e(R)$ is defined to be $\frac{1}{2\pi}$ times the integral over R of the curvature of the metric. It is equal to the 2-cochain that assigns $\frac{1}{2}(2-n)$ to a $2n$-gon region

Motivation

Robert Lipshitz found a way to compute $\mu(\varphi)$ for $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$ in terms of combinatorial data of a Heegaard diagram. This formula is now used everywhere.

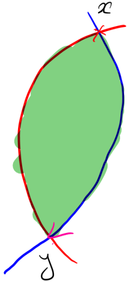
- In the interior of each region R_i of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ pick a point ζ_i
- We put $n_{R_i}(\varphi)$ to be the *intersection number* between φ and $\mathbb{R} \times [0,1] \times \{\zeta_i\}$
- For $p \in \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ define $n_p(\varphi)$ as an average of $n_R(\varphi)$ for those 4 regions to which p belongs
- A *shadow* of φ is a 2-chain $D(\varphi) = \sum_R n_R(\varphi) R$.
- For a region R the Euler measure $e(R)$ is defined to be $\frac{1}{2\pi}$ times the integral over R of the curvature of the metric. It is equal to the 2-cochain that assigns $\frac{1}{2}(2-n)$ to a $2n$-gon region

Theorem (Lipshitz, 2006)

$$\mu(\varphi) = e(D(\varphi)) + n_{\mathbf{x}}(\varphi) + n_{\mathbf{y}}(\varphi)$$

Maslov index formula

1)



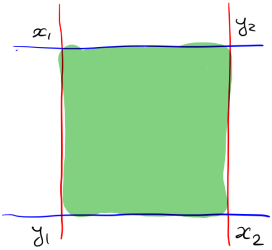
$n_{\bar{x}}(\varphi) =$

$n_{\bar{y}}(\varphi) =$

$e(D(\varphi)) =$

$\mu(\varphi) =$

2)



$n_{\bar{x}}(\varphi) =$

$n_{\bar{y}}(\varphi) =$

$e(D(\varphi)) =$

$\mu(\varphi) =$

Table of Contents

1. Heegaard Floer Homology
2. Maslov index formula
3. Combinatorial proof

Domains

For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ in a Heegaard diagram we denote by $\mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ the set of all 2-chains satisfying $\partial(\partial D \cap \boldsymbol{\alpha}) = \boldsymbol{y} - \boldsymbol{x}$ and $\partial(\partial D \cap \boldsymbol{\beta}) = \boldsymbol{x} - \boldsymbol{y}$. We call any such 2-chain $D \in \mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ a *domain*.

Domains

For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ in a Heegaard diagram we denote by $\mathcal{D}(\boldsymbol{x},\boldsymbol{y})$ the set of all 2-chains satisfying $\partial(\partial D \cap \boldsymbol{\alpha}) = \boldsymbol{y} - \boldsymbol{x}$ and $\partial(\partial D \cap \boldsymbol{\beta}) = \boldsymbol{x} - \boldsymbol{y}$. We call any such 2-chain $D \in \mathcal{D}(\boldsymbol{x},\boldsymbol{y})$ a *domain*.

Introduce a *combinatorial index* of a domain $D \in \mathcal{D}(\boldsymbol{x},\boldsymbol{y})$ via

$$\widetilde{\mu}(D) := n_{\boldsymbol{x}}(D) + n_{\boldsymbol{y}}(D) + e(D)$$

Domains

For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ in a Heegaard diagram we denote by $\mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ the set of all 2-chains satisfying $\partial(\partial D \cap \boldsymbol{\alpha}) = \boldsymbol{y} - \boldsymbol{x}$ and $\partial(\partial D \cap \boldsymbol{\beta}) = \boldsymbol{x} - \boldsymbol{y}$. We call any such 2-chain $D \in \mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ a *domain*.

Introduce a *combinatorial index* of a domain $D \in \mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ via

$$\widetilde{\mu}(D) := n_{\boldsymbol{x}}(D) + n_{\boldsymbol{y}}(D) + e(D)$$

Combinatorial index and Maslov index share several common properties

Domains

For two generators $\mathbf{x}$ and $\mathbf{y}$ in a Heegaard diagram we denote by $\mathcal{D}(\mathbf{x},\mathbf{y})$ the set of all 2-chains satisfying $\partial(\partial D \cap \boldsymbol{\alpha}) = \mathbf{y} - \mathbf{x}$ and $\partial(\partial D \cap \boldsymbol{\beta}) = \mathbf{x} - \mathbf{y}$. We call any such 2-chain $D \in \mathcal{D}(\mathbf{x},\mathbf{y})$ a *domain*.

Introduce a *combinatorial index* of a domain $D \in \mathcal{D}(\mathbf{x},\mathbf{y})$ via

$$\widetilde{\mu}(D) := n_{\mathbf{x}}(D) + n_{\mathbf{y}}(D) + e(D)$$

Combinatorial index and Maslov index share several common properties

1. Additivity of index:

$$\textit{(Sarkar, 2006)} : \widetilde{\mu}(D * D') = \widetilde{\mu}(D) + \widetilde{\mu}(D');$$

$$\mu(\varphi * \varphi') = \mu(\varphi) + \mu(\varphi).$$

Domains

For two generators $\boldsymbol{x}$ and $\boldsymbol{y}$ in a Heegaard diagram we denote by $\mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ the set of all 2-chains satisfying $\partial(\partial D \cap \boldsymbol{\alpha}) = \boldsymbol{y} - \boldsymbol{x}$ and $\partial(\partial D \cap \boldsymbol{\beta}) = \boldsymbol{x} - \boldsymbol{y}$. We call any such 2-chain $D \in \mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ a *domain*.

Introduce a *combinatorial index* of a domain $D \in \mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$ via

$$\widetilde{\mu}(D) := n_{\boldsymbol{x}}(D) + n_{\boldsymbol{y}}(D) + e(D)$$

Combinatorial index and Maslov index share several common properties

1. Additivity of index:

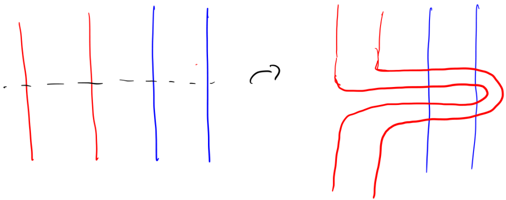
$$\textit{(Sarkar, 2006)} : \widetilde{\mu}(D * D') = \widetilde{\mu}(D) + \widetilde{\mu}(D');$$

$$\mu(\varphi * \varphi') = \mu(\varphi) + \mu(\varphi).$$

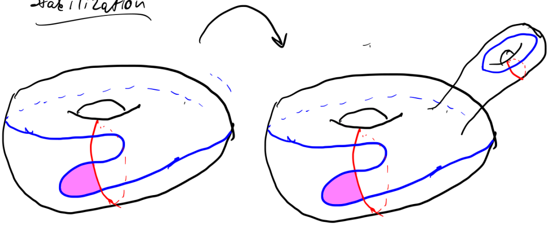
2. *Stability*: index is preserved under *isotopies* and *empty stabilizations* of a Heegaard diagram.

Setup

Isotopy:



Stabilization



Theorem 1

There exists a unique index $\overline{\mu}\colon \mathcal{D} \to \mathbb{Z}$ satisfying the following axioms:

1. $\overline{\mu}$ is additive;
2. $\overline{\mu}$ is stable;
3. $\overline{\mu}(B) = 1$ for any bigon $B \in \mathcal{D}$;
4. $\overline{\mu}(R) = 1$ for any rectangle $R \in \mathcal{D}$.

Moreover, this index agrees with the combinatorial index $\widetilde{\mu}$, and for a Whitney disk φ

$$\mu(\varphi) = \widetilde{\mu}(D(\varphi)).$$

Main results

Theorem 2 (Decomposition)

For a given domain D in a Heegaard diagram (Σ, α, β) there is a sequence of finger moves and empty stabilizations such that in the new Heegaard diagram the image of D can be represented as a composition of bigons, rectangles and their negatives.

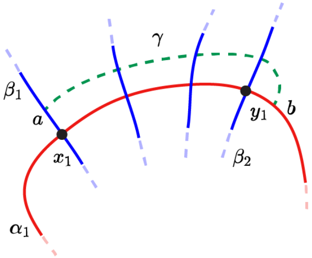
Theorem 2 (Decomposition)

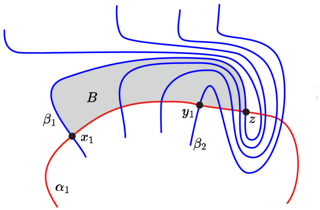
For a given domain D in a Heegaard diagram (Σ, α, β) there is a sequence of finger moves and empty stabilizations such that in the new Heegaard diagram the image of D can be represented as a composition of bigons, rectangles and their negatives.

Proof of Theorem 1

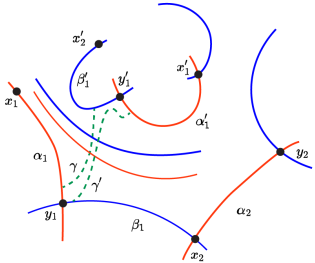
Aplly Theorem 2 to a $D \in \mathcal{D}(\mathbf{x}, \mathbf{y})$ to get some D'. Decompose $D' = D_1 * \ldots * D_k$ into bigons and rectangles. Apply additivity $\overline{\mu}(D) = \overline{\mu}(D_1) + \ldots + \overline{\mu}(D_k)$.

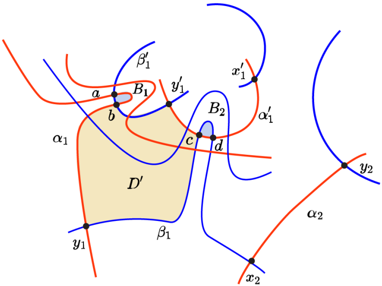
Making the boundary embedded



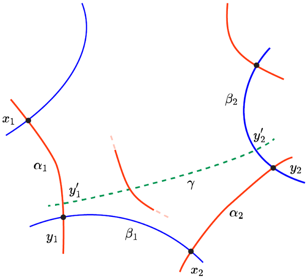


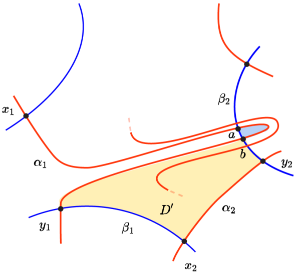
Making the boundary connected



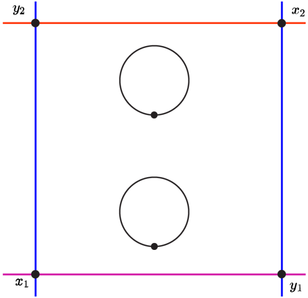


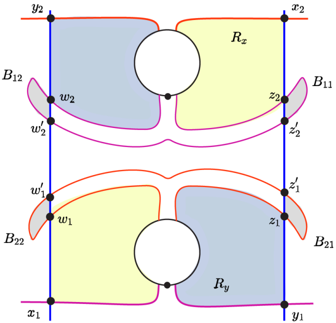
Reducing to the quadrangle or a bigon boundary





Quadrangle boundary





Thank you for your attention!