Global Kuranishi charts for Gromov–Witten moduli spaces and a product formula

Symplectic Zoominar

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Set-up

Let \((X, \omega)\) be a closed symplectic manifold and \(J \in \mathcal{J}_\tau(X, \omega)\). We want to define invariants using the *moduli space of stable maps*

\[
\overline{M}^J_{g, n}(X, \beta).
\]

It consists of \(J\)-holomorphic maps \(u : C \rightarrow X\) where \(C\) is of the form and

1. \(g\) the arithmetic genus,
2. \(n = \#\) marked points,
3. \([u] = \beta \in H_2(X, \mathbb{Z})\),
4. stable iff \(|\text{Aut}(u, C, x_1, \ldots, x_n)| < \infty\).
Gromov–Witten invariants

\( \overline{M}_{g,n}^J(X, \beta) \) is compact and metrisable. Using

\[
\overline{M}_{g,n}^J(X, \beta)
\]

we want to define

\[
GW_{\beta,g,n}^{(X,\omega)} : H^*(X^n, \mathbb{Q}) \to H^*(\overline{M}_{g,n}, \mathbb{Q})
\]

by

\[
GW_{\beta,g,n}^{(X,\omega)}(\alpha) = PD(st_*(ev^*\alpha \cap [\overline{M}_{g,n}^J(X, \beta)]))
\]

**Problem:** \( \overline{M}_{g,n}^J(X, \beta) \) often does not admit a fundamental class in the expected degree \( \rightsquigarrow \) replace with a suitable *virtual fundamental class*
Global Kuranishi chart

A *global Kuranishi chart* $\mathcal{K}$ for a compact space $\mathcal{M}$ consists of

- a compact Lie group $G$,
- a $G$-manifold $\mathcal{T}$, the *thickening*, with finite isotropy
- a $G$-vector bundle $\mathcal{E} \to \mathcal{T}$, the *obstruction bundle*,
- an equivariant section $s: \mathcal{T} \to \mathcal{E}$ such that

$$s^{-1}(0)/G \cong \mathcal{M}.$$ 

The *virtual dimension* of $\mathcal{M}$ is

$$\text{vdim}_\mathcal{K}(\mathcal{M}) := \dim(\mathcal{T}) - \dim(G) - \text{rank}(\mathcal{E}).$$
Virtual fundamental class

An orientation of $\mathcal{K}$ is a $G$-orientation of $[T\mathcal{T} - g]$ and of $\mathcal{E}$.

Given this, the virtual fundamental class $[\mathcal{M}]^{\text{vir}} \in \tilde{H}^d(\mathcal{M}, \mathbb{Q})^*$ is the composition

$$\tilde{H}^{v\dim}(\mathcal{M}; \mathbb{Q}) \xrightarrow{\sim} H^G_{\text{rank}(\mathcal{E})}(\mathcal{T}, \mathcal{T} \backslash s^{-1}(0); \mathbb{Q}) \xrightarrow{s^* \tau} \mathbb{Q}$$

where $\tau$ is the equivariant Thom class of $\mathcal{E}$.

Remark

We use $\mathbb{Q}$-coefficients due to the $G$-action.
Main result

Theorem (H.-Swaminathan, 2022)

Let $g, n \geq 0$ be arbitrary.

a) $\overline{M}_{g,n}(X, \beta)$ admits an oriented global Kuranishi chart of the expected virtual dimension.

b) Its virtual fundamental class is independent of the auxiliary choices made during the construction.

c) Given another $J' \in \mathcal{J}_\tau(X, \omega)$, we can make the auxiliary choices, so that there exists a cobordism between the global Kuranishi charts associated to the respective moduli space.

$$\overset{\sim}{\to} GW_{\beta,g,n}^{(X,\omega)} := PD(st_*(ev^*(-) \cap [\overline{M}_{g,n}(X, \beta)]^{vir}))$$

is well-defined
Remarks

1. *Local* Kuranishi chart has been used in several approaches before. A global Kuranishi chart removes the need for a complicated patching-together of the virtual fundamental class.

2. Abouzaid, McLean and Smith (AMS) were the first to construct a global Kuranishi chart in genus 0. \( \text{(arXiv:2110.14320)} \) They have an independent construction in higher genus.

3. AMS construct a Morava K-theory valued virtual fundamental class \( \rightsquigarrow \) quantum K-theory

4. Bai-Xu: \( \mathbb{Z} \)-valued Gromov–Witten type invariants \( \text{(arXiv:2201.02688)} \)
Theorem (H.-Swaminathan, 2022)

If \((X, \omega) = (X_0, \omega_0) \times (X_1, \omega_1)\), then

\[
\sum_{\beta_i \equiv \beta} GW_{\beta, g, n}^{(X, \omega)} (\alpha_0 \times \alpha_1) = GW_{\beta_0, g, n}^{(X_0, \omega_0)} (\alpha_0) \sim GW_{\beta_1, g, n}^{(X_1, \omega_1)} (\alpha_1)
\]

for any \((g, n) \notin \{(1, 1), (2, 0)\}\) and \(\alpha_i \in H^*(X_i^n, \mathbb{Q})\).

In particular, the small quantum cohomology ring splits over the universal Novikov ring \(\Lambda_0\):

\[
QH^*(X) \cong QH^*(X_0) \otimes_{\Lambda_0} QH^*(X_1).
\]
Constructing a global Kuranishi chart from scratch

\[ \mathcal{M} \rightarrow \widetilde{\mathcal{M}} \xrightarrow{\text{rigidify}} G \xrightarrow{\text{add perturbations to get transversality}} \mathcal{T} \xrightarrow{\text{remove thickening data}} \mathcal{E} + \mathcal{S} \]

\[ \mathcal{M} \] original moduli space

\[ \widetilde{\mathcal{M}} \]

\[ G \]

\[ \mathcal{T} \]

\[ \mathcal{E} + \mathcal{S} \]

\[ \mathcal{B} \]

smooth space of rigidifying data

submersion
In our case (for $n = 0$):

- rigidifying data: highly positive embeddings $C \hookrightarrow \mathbb{P}^N$ (framings);
  \[ \rightsquigarrow \mathcal{B} = \overline{M}_g^*(\mathbb{P}^N, m) \] consists of embedded regular curves in $\mathbb{P}^N$

- perturbations are elements of
  \[ H^0(C, \overline{\text{Hom}}_C(T_{\mathbb{P}^N}|_C, u^* T_X) \otimes \mathcal{O}_C(k)) \otimes H^0(\mathbb{P}^N, \mathcal{O}(k)) \]
  for some $k \gg 1$ (an auxiliary choice)

- $G = \text{PU}(N + 1)$ acts on the framing and the perturbation term

*The case of $n > 0$ follows formally.*
Equivalence of global Kuranishi charts

We call two global Kuranishi charts *equivalent* if they are related by a zigzag of the following moves

- **(Germ equivalence)** Given $U \subseteq \mathcal{T}$ open $G$-invariant neighbourhood of $s^{-1}(0)$: $(G, \mathcal{T}, \mathcal{E}, s) \leadsto (G, U, \mathcal{E}|_U, s|_U)$

- **(Group enlargement)** Given a principal $G'$-bundle $q: \mathcal{P} \rightarrow \mathcal{T}$ with compatible $G$-action: $(G, \mathcal{T}, \mathcal{E}, s) \leadsto (G \times G', \mathcal{P}, q^*\mathcal{E}, q^*s)$

- **(Stabilisation)** Given a $G$-vector bundle $p: \mathcal{W} \rightarrow \mathcal{T}$: $(G, \mathcal{T}, \mathcal{E}, s) \leadsto (G, \mathcal{W}, p^*\mathcal{E} \oplus p^*\mathcal{W}, p^*s \oplus \delta_\mathcal{W})$
Constructing a global Kuranishi chart from a given one

**Question:** Given a map $\psi: \mathcal{M} \rightarrow \mathcal{N}$ between moduli spaces, can we lift it to a map $\mathcal{K}_{\mathcal{M}} \rightarrow \mathcal{K}_{\mathcal{N}}$ between global Kuranishi charts?

**Answer:** Usually not.

**But:** Global Kuranishi charts can be pulled back along maps to their base space.

$\leadsto$ use $\mathcal{K}_{\mathcal{N}}$ to construct a new global Kuranishi chart $\tilde{\mathcal{K}}_{\mathcal{M}}$ which

i) comes with a natural map $\tilde{\mathcal{K}}_{\mathcal{M}} \rightarrow \mathcal{K}_{\mathcal{N}}$

ii) is equivalent to $\mathcal{K}_{\mathcal{M}}$ (work required).

**Remark:** This is the key idea in the proof of the product formula, using the map

$$\overline{M}_{g,n}^* (\mathbb{P}^{N_0} \times \mathbb{P}^{N_1}, (m_0, m_1)) \rightarrow \overline{M}_{g,n}^* (\mathbb{P}^{N_0}, m_0) \times \overline{M}_{g,n}^* (\mathbb{P}^{N_1}, m_1).$$
Thank you for your attention!