Locally maximizing orbits and rigidity for Convex billiards

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Symplectic
Zoominar

Based on joint works with
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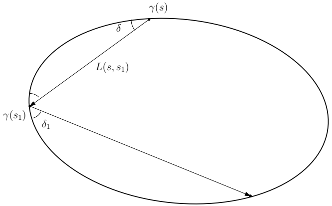
Twist maps

Consider an exact symplectic twist map T of the cylinder $\mathbf{A}$ with symplectic coordinates (q, p). Denote by $H(q, q')$ the generating function, i.e $\boxed{p'dq' - pdq = dH}$ so that $\boxed{p' = \partial_2 H(q, q'), p = -\partial_1 H(q, q')}$. We assume the negative twist condition: $H_{12}(q, q') > 0$. Variational principle for configurations: $\{q_n\} \to \sum_n H(q_n, q_{n+1})$.

Example

1. The standard-like map $H(q, q') = \frac{1}{2}(q - q')^2 + V(q)$, where V is a periodic potential. Here $H_{12} = -1$- positive twist.
2. Let γ be a C^2-convex closed curve in the plane. Birkhoff billiard map $\boxed{T : (s, \delta) \mapsto (s_1, \delta_1) \ \Leftrightarrow \ L_s = -\cos\delta, L_{s_1} = \cos\delta_1}$. Here $L(s, s_1) = |\gamma(s) - \gamma(s_1)|$, and $L_{12} = \frac{\sin\delta\sin\delta_1}{L} > 0$-negative twist.



We study *locally maximizing* configurations, i.e. those configurations $\{q_n\}$ such that any finite subsegment $\{q_n\}_{n=M}^{N}, M \leq N$ is a local maximum of the truncated functional

$$F_{M,N}(x_M, ..., x_N) = H(q_{M-1}, x_M) + \sum_{i=M}^{N-1} H(x_i, x_{i+1}) + H(x_N, q_{N+1}).$$

We shall call such configurations, m-configurations, and the corresponding orbits on the phase cylinder $\mathbf{A}$, m-orbits.
If the matrix of second variation of some finite segment of a configuration $\{q_n\}$ is negative semi-definite, then the matrix of second variation of any proper sub-segment is negative definite.
Let $\mathcal{M}_H \subset \mathbf{A}$ the set swept by all *m-orbits* corresponding to H. The set $\mathcal{M}_H$ is a closed invariant set of T.
By twist map theory, $\mathcal{M}_H$ contains all rotational invariant curves as well as Aubry-Mather sets (Cantor-tori).

Criterion for m-orbits

Theorem 1 [B.-Tsodikovich]. *Let $T : \mathbf{A} \mapsto \mathbf{A}$ be an exact twist map with the twist condition $H_{12} > 0$. Then the orbit $\{(q_n, p_n)\}$ is an m-orbit if and only if there exists a positive Jacobi field along $\{q_n\}$.*

A *Jacobi field* along a configuration $\{q_n\}$ is a sequence $\{\delta q_n\}$ satisfying the *discrete Jacobi equation*:

$$b_{n-1}\delta q_{n-1} + a_n \delta q_n + b_n \delta q_{n+1} = 0, \tag{1}$$

$$a_n := H_{22}(q_{n-1}, q_n) + H_{11}(q_n, q_{n+1}),\ \ b_n := H_{12}(q_n, q_{n+1}).$$

Any solution to the Jacobi equation $\{\delta q_n\}$ can be lifted to a T-invariant vector field $(\delta q_n, \delta p_n)$ along the orbit (and vice versa) where

$$\delta p_n = -H_{11}(q_n, q_{n+1})\delta q_n - H_{12}(q_n, q_{n+1})\delta q_{n+1},$$

or equivalently, due to the Jacobi equation:

$$\delta p_n = H_{22}(q_{n-1}, q_n)\delta q_n + H_{12}(q_{n-1}, q_n)\delta q_{n-1}.$$

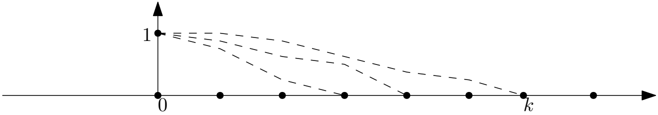
The matrix of second variation of F_{MN} has the following Jacobi form:

$$W_{MN} = \delta^2 F_{MN} = \begin{pmatrix} a_M & b_M & 0\cdots & 0 & 0 \\ b_M^* & a_{M+1} & b_{M+1}\cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{N-2}^* & a_{N-1} & b_{N-1} \\ 0 & \cdots & 0 & b_{N-1}^* & a_N \end{pmatrix},$$

Proof of the criterion uses *discreet Sturm theory* for the Jacobi equation. For example positive Jacobi field can be constructed (similar to E.Hopf in Riemannian case) as follows:

Set $\xi_n^{(k)}$ be the Jacobi field satisfying the boundary conditions

$$\boxed{\xi_0^{(k)} = 1, \xi_k^{(k)} = 0, \text{ and define } \delta q_n := \lim_{k\to+\infty} \xi_n^{(k)} \Rightarrow \delta q_n > 0}$$



Here one can show that the graphs of two Jacobi fields intersect at most once (due to negative definiteness of W).

Criterion in the opposite direction

Assume now that there exists a positive Jacobi field along a configuration $\{q_n\}$. Then it follows from the discrete Sturm Separation Theorem that any other Jacobi field along $\{q_n\}$ which vanishes at $n = K$ keeps a constant sign for all $n < K$ and the opposite sign for all $n > K$. We show that the matrix W_{1N} is negative definite. For the principal minors M_k of the matrix W_{1N} we have the recursion formula

$$M_{k+1} = a_{k+1} M_k - b_k^2 M_{k-1},$$

where by convention $M_0 = 1, M_{-1} = 0$.
The Jacobi field $\{\xi_n\}$ such that $\xi_0 = 0$, $\xi_1 = 1$. Then we have the formula

$$\xi_{k+1} = (-1)^k \frac{M_k}{b_1 b_2 \cdots b_k}. \quad (2)$$

Indeed, (2) holds true for $k = 1$, and then can be verified by induction. It follows from (2) that the sign of M_k equals $(-1)^k$ for all $k \geq 1$, since all ξ_k are positive for $k \geq 1$. This proves negative definiteness of the matrix W_{1N}.

Function ω

Let $\{(p_n, q_n)\}$ be an m-orbit of the point $z = (p_0, q_0)$. Then by the Criterion there exists an invariant vector field $\{(\delta p_n, \delta q_n)\}$ along the orbit $\{(p_n, q_n)\}$ such that $\delta q_n > 0$ (normalized by $\delta q_0 = 1$).

$$\delta p_n = -H_{11}(q_n, q_{n+1})\delta q_n - H_{12}(q_n, q_{n+1})\delta q_{n+1},$$

or equivalently, due to the Jacobi equation:

$$\delta p_n = H_{22}(q_{n-1}, q_n)\delta q_n + H_{12}(q_{n-1}, q_n)\delta q_{n-1}.$$

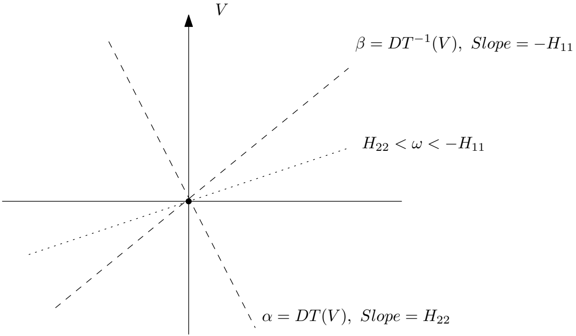
Set $\boxed{\omega(p_n, q_n) := \frac{\delta p_n}{\delta q_n}}$. One can prove that ω is measurable on $\mathcal{M}$ and

$$\begin{cases} \omega(T(p,q)) = H_{22}(q, q_1) + H_{12}(q, q_1)\delta q_1(q,p)^{-1}, \\ \omega(p,q) = -H_{11}(q, q_1) - H_{12}(q, q_1)\delta q_1(q,p). \end{cases}$$

Therefore by the twist condition we have the bounds

$$H_{22}(q_{-1}, q_0) < \omega(q_0, p_0) < -H_{11}(q_0, q_1). \quad (3)$$

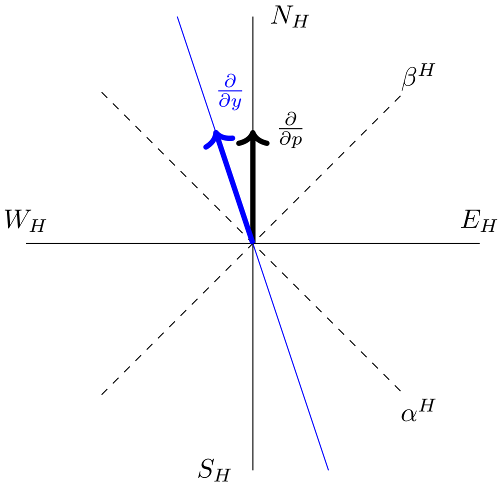
(used by MacKay, Percival for converse KAM). Here H_{22} (and $-H_{11}$) is the slope of the image (respectively pre-image) of $\partial/\partial p$ under T.


V
$\beta = DT^{-1}(V),\ Slope = -H_{11}$
$H_{22} < \omega < -H_{11}$
$\alpha = DT(V),\ Slope = H_{22}$

Two generating functions

Consider a map of the cylinder $\mathbf{A}$ which is a twist map wrt two sets of symplectic coordinates (q, p) and (x, y), with generating functions H, G respectively.
Natural question: Does the equality $\mathcal{M}_H = \mathcal{M}_G$ hold ? We claim "yes" under assumption (GA) below.
At every point $z \in \mathcal{M}_H$ one can partition the tangent space $T_z\mathbf{A}$ to four cones that are determined by the image α and pre-image β (by T) of the vertical direction $\frac{\partial}{\partial p}$.
We denote by N_H the "north" cone. Similarly, we define N_G, for every point $z \in \mathcal{M}_G$. by the image and pre-image of the vertical direction $\frac{\partial}{\partial y}$.
We shall assume the following:

$$\textbf{Geometric assumption :}\ \begin{cases} \forall z \in \mathcal{M}_H \implies \frac{\partial}{\partial y}(z) \in N_H, \\ \forall z \in \mathcal{M}_G \implies \frac{\partial}{\partial p}(z) \in N_G. \end{cases} \qquad (4)$$



Theorem 2 [B.-Tsodikovich]. *Let $T : \mathbf{A} \to \mathbf{A}$ be an exact twist map, with respect to two sets of symplectic coordinates, (q, p) and (x, y) and generating functions H, G satisfying the twist condition $H_{12}, G_{12} > 0$. Assume the geometric assumption holds. Then the sets of m-orbits corresponding to the variational principles for H and G coincide:*

$$\mathcal{M}_H = \mathcal{M}_G.$$

Corollary Let L, S be generating functions for billiard map, as we explain below. Then Geometric assumption is satisfied (can be checked explicitly) and hence $\mathcal{M}_L = \mathcal{M}_S$.

Remark Theorem and Corollary hold true for higher dimensions, Geometric assumption must be modified slightly: there exists a homotopy of Lagrangian subspaces $\{V_t\}$ connecting V^S and V^H, such that the subspace V_t is transversal to all four subspaces α^S, β^S, α^H, β^H.

GA versus Tonelli

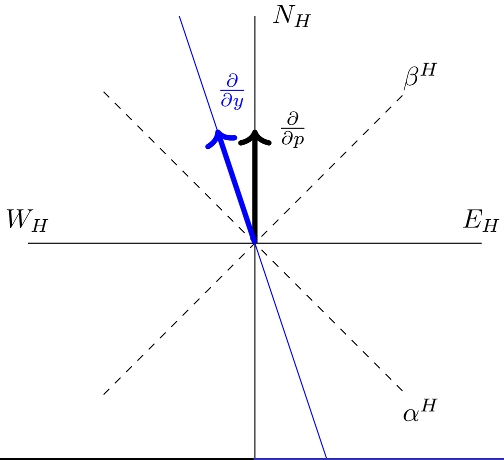
Let us remind a result by Bernard and by Mazzucchelli and Sorentino on Tonelli Hamiltonians. It was shown that if a Tonelli Hamiltonian remains Tonelli after an exact symplectic change of variables, then the Aubry, Mañe, and Mather sets are the same for both Hamiltonians.

Every exact symplectic twist map of a cylinder can be seen as a time-one map for some Tonelli Hamiltonian by Moser, 1986. This suggests that in the two dimensional case, the result on Tonelli Hamiltonians and our results may be connected. It is also known that a higher dimensional twist maps that satisfies a "symmetric" twist condition is also a time-one map of a Tonelli Hamiltonian.

However, this interpolation result is not known for general Twist maps/ Birkhoff billiards in high dimension. In our approach we deal directly with the discrete system, and do not rely on interpolation by Tonelli Hamiltonians.

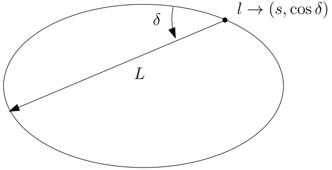
Proof of Theorem 2.

Let us show, for example, the inclusion $\mathcal{M}_H \subseteq \mathcal{M}_G$. Take any m-orbit (q_n, p_n) in $\mathcal{M}_H$. By the Criterion, there exists a non-vertical T-invariant vector field $(\delta q_n, \delta p_n)$ along the m-orbit, with $\delta q_n > 0$. Since for $\omega = \frac{\delta p_n}{\delta q_n}$ we know $H_{22} < \omega < -H_{11}$, then this vector field lies in the cone E_H. Hence this invariant field in coordinates (x, y) satisfies $\delta x_n > 0$.

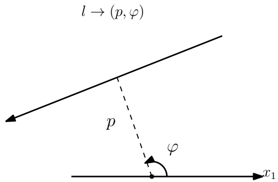


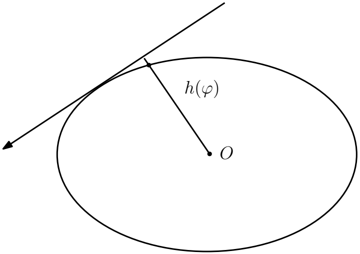
Symplectic structure on the space of lines.

Phase cylinder $\mathbf{A}$=Space of oriented lines. Natural symplectic structure ω can be written in two ways:



$$\omega = d(\cos\delta ds) = \sin\delta ds \wedge d\delta$$





$$\omega = dp \wedge d\varphi$$

$$x_1 \cos\varphi + x_2 \sin\varphi = h(\varphi)$$

Generating function L

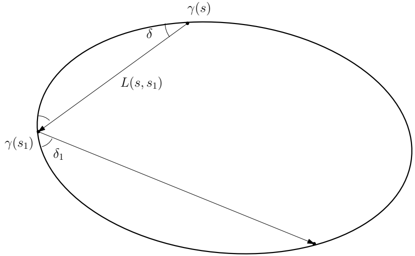
Let T be the billiard map $T : \mathbf{A} \to \mathbf{A},\ T^*\omega = \omega$.
For the primitive 1-form $\lambda_1 = (\cos\delta)ds,\ \ d\lambda_1 = \omega$ we have

$$T^*\lambda_1 - \lambda_1 = dL \quad \Leftrightarrow \quad \cos\delta_1 ds_1 - \cos\delta ds = dL(s, s_1).$$

Here

$$T : (s, \delta) \mapsto (s_1, \delta_1) \ \Leftrightarrow\ L_s = -\cos\delta, L_{s_1} = \cos\delta_1.$$

Here $L(s, s_1) = |\gamma(s) - \gamma(s_1)|$ the length of the chord.



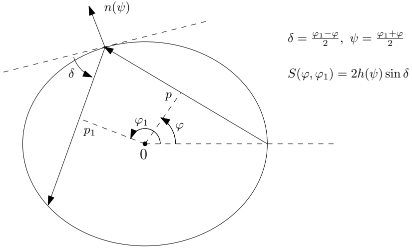
Non-standard generating function S in the plane

$\omega = dp \wedge d\varphi,\ \lambda_2 = pd\varphi.$

$$T^*\lambda_2 - \lambda_2 = p_1 d\varphi_1 - pd\varphi = dS$$

Theorem[B-M] *For planar billiard Generating function S takes the form*

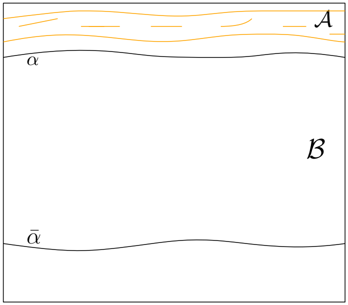
$$\boxed{S(\varphi,\varphi_1) = 2h\left(\psi\right)\sin\left(\delta\right);\ \psi = \frac{\varphi+\varphi_1}{2},\ \delta = \frac{\varphi_1-\varphi}{2}}$$



$$p = h(\psi)\cos\delta - h'(\psi)\sin\delta = -S_\varphi$$
$$p_1 = h(\psi)\cos\delta + h'(\psi)\sin\delta = S_{\varphi_1}$$

Effective version of Birkhoff conjecture

Non-standard generating function S can be used to give an effective version of Birkhoff conjecture for centrally symmetric curves. This is done by sharp estimation of the invariant set Δ which is the complement to the set $\mathcal{M}$ occupied by locally-maximizing orbits (m-$orbits$). Set $\mathcal{M}$ contains all rotational invariant curves and all Cantor tori.
It is crucial that the set $\mathcal{M}$ does not depend on the choice of the generating function L or S. Let α be the invariant curve of rotation number $1/4$ consisting of 4-periodic orbits.



$\mathbf{A}$

$$\Delta = \mathbf{A} - \mathcal{M}$$

$$\Delta_{\mathcal{A}} = \mathcal{A} - \mathcal{M}, \quad \Delta_{\mathcal{B}} = \mathcal{B} - \mathcal{M}$$

Consider the *class* $\mathcal{C}$ *of centrally symmetric, strictly convex,* C^2 *smooth curves*, for which the billiard map has an invariant curve α with rotation number $1/4$ consisting of 4-periodic orbits.
Let $\mathcal{A}$ be the domain bounded by α and the upper boundary of the cylinder, and by $\mathcal{M}$ the subset of $\mathbf{A}$ swept by m-orbits.

Theorem 3 [B.-Tsodikovich]. *Let $\gamma \in \mathcal{C}$, Denote by $h : [0, 2\pi] \to \mathbf{R}$ the support function of γ, with respect to the center of symmetry. Set $\Delta_{\mathcal{A}} = \mathcal{A} \setminus \mathcal{M}$. Then the following estimate for the measure holds true:*

$$\mu(\Delta_{\mathcal{A}}) \geq \frac{25\pi^2}{32} \beta^3 d^2(h^2, U),$$

where $0 < \beta$ is the minimal curvature of γ, U is the subspace of $L^2[0, \pi]$ spanned by $\{1, \cos(2\psi), \sin(2\psi)\}$, and $d(\cdot, U)$ is the L^2- distance from this subspace. Moreover, this bound is sharp for ellipses.

Birkhoff conjecture for centrally symmetric case [B-M]

Corollaries of Theorem 3

1. If $\mu(\Delta_{\mathcal{A}}) = 0$ then γ is an ellipse.

2. Assume the domain $\mathcal{A}$ is foliated by rotational invariant curves of T. Then by M.Hermann' result each orbit is m-orbit and hence it follows from 1. that γ is an ellipse.

3. Assume that there is a field of non-vertical lines on $\mathcal{A}$, oriented to the right, and such that it is invariant under T (together with the orientation) then γ is an ellipse (Every orbit is an m-orbit, since there exists a positive Jacobi field, hence Theorem 3 applies).

4. For $\gamma \in \mathcal{C}$ which is not ellipse, there always exist billiard orbits in $\mathcal{A}$ having conjugate points with respect to each set of coordinates (s, δ) and (φ, p).

Effective Rigidity apart from the boundary

Let $\gamma \in \mathcal{C}$, let $\alpha, \bar{\alpha}$ be the invariant curves of 4-periodic orbits, $\mathcal{B}$ be the domain between them on the phase cylinder.
Let $\Delta_{\mathcal{B}} := \mathcal{B} \setminus \mathcal{M}$.

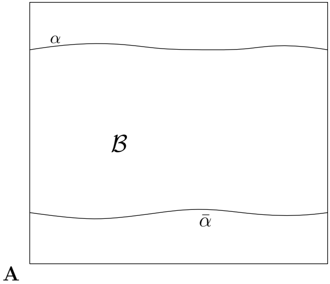


Figure: The region $\mathcal{B}$

Theorem 4 [B]. *Suppose that the billiard ball map T of γ has a continuous rotational invariant curve $\alpha \subset \mathbf{A}$ of rotation number $1/4$, consisting of 4-periodic orbits. Let $\bar{\alpha}$ be the corresponding invariant curve of rotation number $\frac{3}{4}$. Then the following estimate holds;*

$$\frac{3}{64}\beta(P^2 - 4\pi A) \leq \mu(\Delta_{\mathcal{B}}), \tag{5}$$

where P,A denote the perimeter and the area of γ, and β is the minimal curvature of γ.

Corollary

If $\mu(\Delta_{\mathcal{B}}) = 0$ then γ is a circle.

Mather β-function

Given a rotation number ρ, Mather β-function for a positive twist map assigns to ρ the average action of a minimal action trajectory with rotation number ρ. For convex billiard $\beta(\frac{m}{n})$ = perimeter of the periodic billiard configuration divided by the number of vertices.

Theorem 5 [B]. *Let Ω_1, Ω_2 be two strictly convex C^2-smooth centrally symmetric planar domains such that Ω_1 is an ellipse. Suppose that Mather β-functions β_1, β_2 of the domains satisfy*

$$\boxed{\beta_1(\rho) = \beta_2(\rho),\ \forall \rho \in \left(0, \frac{1}{4}\right].}$$

Then Ω_2 is an ellipse isometric to Ω_1.

Remarks

1. Can other domains be distinguished by the β-function.
2. This circle of questions was first addressed in the book by K.F.Siburg, then by Kaloshin, Sorrentino.

β-function of ellipses explicitly

There is a simple way to compute β-function for ellipses using non-standard generating function.

Theorem (...?;Reznik, Garcia, Koiller; B.)

Consider the invariant curve of rotation number ρ corresponding to the caustic $E_\lambda = \left\{ \frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} = 1 \right\}, \quad 0 < \lambda < b^2 < a^2$. Then

$$\boxed{\beta(\rho) = \frac{2ce\sqrt{e^2 - f^2}}{e^2 - 1} - \frac{2cf}{K(k)}[K(k)E(\phi, k) - E(k)F(\phi, k)],}$$ *where*

$k = \frac{1}{f},\ \phi = \arcsin \frac{\sqrt{\lambda}}{b}$*, and* $E(\phi, k)$ *is elliptic integral of the second kind,* $K(k), E(k)$ *are complete elliptic integrals of first and second kind, and* e, f *are eccentricities of the ellipses* E, E_λ*.*

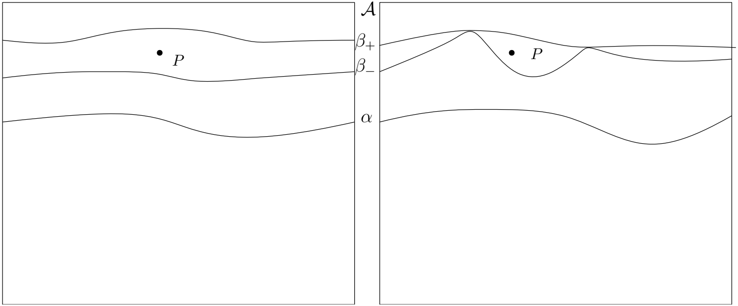
Corollary

$$\boxed{\beta(\rho) = \frac{2a\sqrt{\lambda}}{b} - 2\sqrt{a^2 - \lambda}E(\phi, k) + \rho|E_\lambda|,}$$ *where*

$\phi = \arcsin \frac{\sqrt{\lambda}}{b},\ k = 1/f$*, and* $|E_\lambda|$ *is the circumference of* E_λ*.*

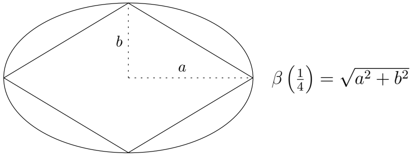
Proof of Theorem 5

1. By a Theorem of J.Mather the function β is differentiable at any irrational point. It is differentiable at a rational point ρ, iff there is a rotational invariant curve consisting of periodic orbits with rotation number ρ. It then follows from this theorem and Aubry-Mather theory that there exist invariant curves of all rotation numbers $\rho \in (0, \frac{1}{4}]$ which foliate the annulus $\mathcal{A}$. Thus **Corollary 1** implies that Ω_2 must be an ellipse.



2. We need to show that this ellipse is an isometric copy of Ω_1. Indeed let $a_i > b_i, i = 1, 2$ are the semi-axis. First,

$$\beta_1\Big(\frac{1}{4}\Big) = \beta_2\Big(\frac{1}{4}\Big) \Rightarrow a_1^2 + b_1^2 = a_2^2 + b_2^2.$$



Second, mention that by the definition $\beta(0) = 0$ holds true for any domain. But, $\beta_1'(0) = |\Omega_1| = \beta_2'(0) = |\Omega_2|$ -the circumferences of the domains. We have:

$$|\Omega| = \int_0^{2\pi} h(\psi) d\psi.$$

Therefore for the ellipses $\Omega_{1,2}$ we write

$$|\Omega_i| = 4\int_0^{\pi/2} \sqrt{\frac{a_i^2+b_i^2}{2} + \frac{a_i^2-b_i^2}{2}\cos 2\psi} \quad d\psi =$$

$$= 2\sqrt{2}\int_0^{\pi} \sqrt{(a_i^2+b_i^2)+(a_i^2-b_i^2)\cos t}\ dt = 2\sqrt{2}\int_0^{\pi}\sqrt{A + c_i^2\cos t}\ dt,$$

where $A := a_1^2 + b_1^2 = a_2^2 + b_2^2$. Consider now the last integral as a function of the parameter $C := c^2 = a^2 - b^2$, while A is fixed.

$$f(C) := 2\sqrt{2}\int_0^{\pi}\sqrt{A + C\cos t}\ dt$$

Differentiating f with respect to C we obtain for $f'(C)$:

$$\sqrt{2}\int_0^{\pi}\frac{\cos t}{\sqrt{A + C\cos t}}\,dt = \sqrt{2}\int_0^{\pi/2}\left[\frac{\cos t}{\sqrt{A + C\cos t}} - \frac{\cos t}{\sqrt{A - C\cos t}}\right] dt.$$

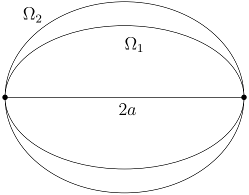
It is easy to see that the for $t \in (0, \pi/2)$ the integrand is negative, hence f is strictly monotone decreasing in C. Therefore, the equality $|\Omega_1| = |\Omega_2|$ is possible only when $C_1 = C_2$. Hence the ellipses are isometric. $\square$

Question. *How many values of β-function determine the ellipse in the class of ellipses. More precisely we ask if ellipse is determined by any two values of β-function $\beta(\rho_1), \beta(\rho_2)$ for the rotation numbers $\rho_{1,2} \in (0, \frac{1}{2}]$.*
Notice that in [Sorrentino] the reconstruction of ellipse is given by means of infinitesimal data of the β-function near 0.
A partial result in the direction of this question is the following

Theorem 6. *Ellipse can be determined by two values of $\beta(\rho_1), \beta(\rho_2)$ where $\rho_1 = \frac{1}{2}$ and $\rho_2 = \frac{m}{n}$ is any rational in $(0, \frac{1}{2})$.*

Proof
Notice first that $\beta(\frac{1}{2}) = 2a$ is the diameter of ellipse. We argue by contradiction. Suppose Ω_1, Ω_2 are two ellipses with the same diameter $2a$, satisfying $\beta_1(\frac{m}{n}) = \beta_2(\frac{m}{n})$, but $b_1 < b_2$.

Introduce a linear map A which is the expansion map along the y-axes transforming Ω_1 to Ω_2.



Denote by P_1, P_2 two Poncelet polygons of the rotation number $\frac{m}{n}$ for Ω_1 and Ω_2, respectively. Obviously, the polygons $A(P_1)$ and P_2 have the same rotation number. The condition $\beta_1(\frac{m}{n}) = \beta_2(\frac{m}{n})$ implies that the perimeters of $P_{1,2}$ are equal:

$$|P_1| = |P_2| \Rightarrow |A(P_1)| > |P_2|.$$

However, this contradicts the fact P_2 is a Poncelet polygon is a length maximizer in its homotopy class. $\square$

Remark. It is plausible that the result remains valid when the rotation number ρ_2 is irrational.

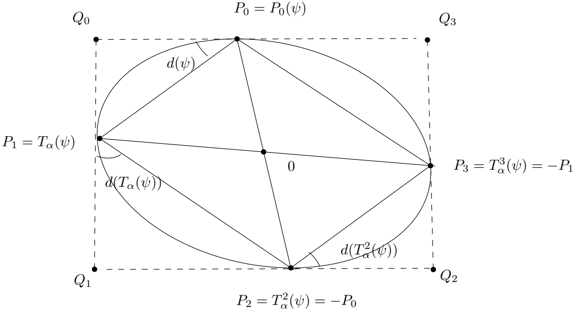
Idea of Proof of Theorem 3.

Let $\mathcal{C}$ be the class of centrally symmetric, strictly convex, C^2 smooth curves, for which the billiard map has an invariant curve of 4-periodic orbits.

Proposition. *For the class $\mathcal{C}$ the following description holds:*

$$\mathcal{C} = \{\gamma \mid h^2(\psi) = c_0 + \sum_{n\in 2+4\mathbf{Z}} c_n e^{in\psi}, h + h'' > 0\}. \tag{6}$$

More precisely, given a C^2 function $h : [0, 2\pi] \to \mathbf{R}$ satisfying both conditions in the definition of the class $\mathcal{C}$, there exists a curve $\gamma \in \mathcal{C}$ for which the support function is h.



Fourier decomposition of h^2 also yields the identity $h^2(\psi) + h^2(\psi + \frac{\pi}{2}) = 2c_0 =: R^2$, where $c_0 = h^2(0)$. Therefore, we can find a π-periodic function $d, 0 < d(\psi) < \frac{\pi}{2}$ for which

$$\begin{cases} h(\psi) = R \sin d(\psi), \\ h(\psi + \frac{\pi}{2}) = R \cos d(\psi). \end{cases} \tag{7}$$

Theorem 2 implies that we can can work with the generating function S. Consider an m-orbit of the point (φ, p).

We have the relations:

$$\begin{cases} \omega(p_1, \varphi_1) = S_{22}(\varphi, \varphi_1) + S_{12}(\varphi, \varphi_1)\delta\varphi_1(\varphi, p)^{-1}, \\ \omega(p, \varphi) = -S_{11}(\varphi, \varphi_1) - S_{12}(\varphi, \varphi_1)\delta\varphi_1(\varphi, p). \end{cases}$$

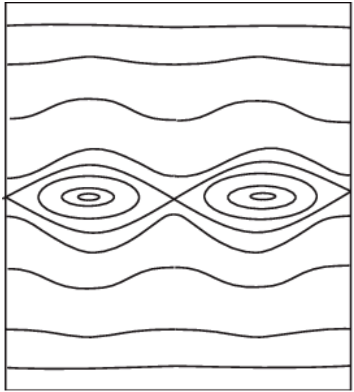
We multiply the first equation by p_1^2, and the second by p^2, and subtract

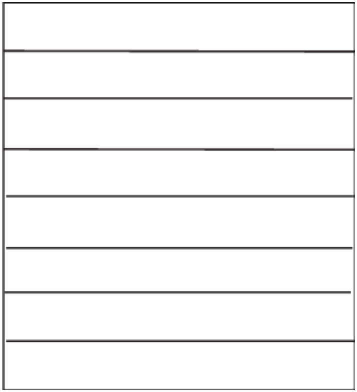
$$p_1^2\omega(\varphi_1, p_1) - p^2\omega(\varphi, p) =$$
$$= p^2 S_{11}(\varphi, \varphi_1) + p_1^2 S_{22}(\varphi, \varphi_1) + S_{12}\Big(p^2\delta\varphi_1(\varphi, p) + p_1^2\delta\varphi_1(\varphi, p)^{-1}\Big) \geq$$
$$\geq p^2 S_{11}(\varphi, \varphi_1) + p_1^2 S_{22}(\varphi, \varphi_1) + 2pp_1 S_{12}(\varphi, \varphi_1).$$

We used the fact that $\delta\varphi_1$ and S_{12} are positive. Now integrate both sides of this inequality on the invariant set $\mathcal{M} \cap \mathcal{A}$, with respect to the invariant measure $d\mu$. The integral of the left hand side vanishes, hence:

$$\int_{\mathcal{M}\cap\mathcal{A}} \Big(p^2 S_{11}(\varphi, \varphi_1) + p_1^2 S_{22}(\varphi, \varphi_1) + 2pp_1 S_{12}(\varphi, \varphi_1)\Big) d\mu \leq 0.$$

Ellipse vs circle





Substitute the second partials of S and p, p_1 and
$d\mu = \frac{1}{4}(h(\psi) + h''(\psi)) \sin\delta d\delta d\psi$.
$p = -S_1(\varphi, \varphi_1) = h(\psi)\cos\delta - h'(\psi)\sin\delta$
$p_1 = S_2(\varphi, \varphi_1) = h(\psi)\cos\delta + h'(\psi)\sin\delta,$
where $\psi = \frac{\varphi+\varphi_1}{2}$, $\delta = \frac{\varphi_1-\varphi}{2}$.
Then after simplification, we get the following inequality

$$0 \geq \int\limits_{\mathcal{M}\cap\mathcal{A}} \left[\cos^2\delta\sin\delta\Big(h''h^2 + 3h(h')^2\Big) - h(h')^2\sin\delta\right] d\mu. \qquad (8)$$

Call the first summand of the integrand A, and the second one B. Then inequality (8) gives:

$$0 \geq \int\limits_{\mathcal{M}\cap\mathcal{A}} A - B\, d\mu = \int\limits_{\mathcal{A}} A - B\, d\mu - \int\limits_{\Delta_{\mathcal{A}}} A - B\, d\mu \geq \int\limits_{\mathcal{A}} A - B\, d\mu - \int\limits_{\Delta_{\mathcal{A}}} A\, d\mu,$$

Since the function $B = h(h')^2\sin\delta$ is non-negative.

Therefore we get:

$$\int_{\mathcal{A}} (A - B) d\mu \leq \int_{\Delta_{\mathcal{A}}} A d\mu. \tag{9}$$

Next, it can be simplified by Lemma 5.1 of [Bialy-Mironov]

$$\int_{\mathcal{A}} (A - B) d\mu = \frac{\pi R^4}{1024} \int_0^{\pi} (\mu'')^2 - 4(\mu')^2 d\psi,$$

where $\mu(\psi) = \cos(2d(\psi))$. Now we bound this integral from below, and bound $\int_{\Delta_{\mathcal{A}}} A d\mu$ from above. It holds that

$$\mu(\psi) = \cos(2d(\psi)) = 1 - 2\sin^2 d(\psi) = 1 - 2\frac{h^2(\psi)}{R^2}.$$

As a result, $\mu'(\psi) = -\frac{2}{R^2}(h^2)'$, and $\mu''(\psi) = -\frac{2}{R^2}(h^2)''$.

Thus, we have

$$\int_0^{\pi} (\mu'')^2 - 4(\mu')^2 d\psi = \frac{4}{R^4} \int_0^{\pi} \Big((h^2)''\Big)^2 - 4\Big((h^2)'\Big)^2 d\psi.$$

Since γ is a curve in $\mathcal{C}$, the Fourier expansion of h^2 is as in equation (6). Now use Parseval's identity in $L^2[0,\pi]$:

$$\int_0^{\pi} \Big((h^2)''\Big)^2 - 4\Big((h^2)'\Big)^2 d\psi = \pi \sum_{n \in 2+4\mathbf{Z}} (n^4 - 4n^2)|c_n|^2 =$$

$$= \pi \sum_{\substack{n \in 2+4\mathbf{Z} \\ |n| > 2}} (n^4 - 4n^2)|c_n|^2 \geq \pi \sum_{\substack{n \in 2+4\mathbf{Z} \\ |n| > 2}} 1000|c_n|^2,$$

since for $|n| \geq 6$, $n^4 - 4n^2 \geq 1000$. By Parseval's identity again

$$\int_0^{\pi} \Big((h^2)''\Big)^2 - 4\Big((h^2)'\Big)^2 d\psi \geq 1000 \int_0^{\pi} |h^2 - c_0 - c_2 e^{2i\psi} - c_{-2} e^{-2i\psi}|^2 d\psi \geq$$

$$\geq 1000\pi d^2(h^2, \mathrm{Span}\{1, \cos(2\psi), \sin(2\psi)\}),$$

d denotes the distance in the L^2 norm between the function h^2 and a subspace of $L^2[0,\pi]$. Denote the subspace of $L^2[0,\pi]$ spanned by $\{1, \cos(2\psi), \sin(2\psi)\}$ by U. Thus we proved

$$\int_0^\pi (\mu'')^2 - 4(\mu')^2 d\psi \geq \frac{4000\pi}{R^4} d^2(h^2, U).$$

As a result, we get the following lower bound:

$$\int_{\mathcal{A}} (A - B) d\mu \geq \frac{\pi R^4}{1024} \cdot \frac{4000\pi}{R^4} d^2(h^2, U) = \frac{125\pi^2}{32} d^2(h^2, U). \quad (10)$$

Now we turn to finding an upper bound for $\int_{\Delta_{\mathcal{A}}} A d\mu$. If N is an upper bound on A, then

$$\int_{\Delta_{\mathcal{A}}} A d\mu \leq N\mu(\Delta),$$

so it is enough to find an upper bound for A.

$$|A| = |\sin\delta\cos^2\delta(h''h^2 + 3h(h')^2)| \le |h''|h^2 + 3h(h')^2.$$

Since γ is centrally symmetric, $h(\psi)$ is half the width in the direction ψ. The maximal width is in the direction of the diameter, so by Blaschke rolling disc theorem, we get $\boxed{h \le \frac{D}{2} \le \frac{1}{\beta}.}$ Next, since $h + h'' = \rho$, then $\boxed{|h''| \le \rho + h \le \frac{1}{\beta} + \frac{D}{2} \le \frac{2}{\beta}}$, where β is the minimal curvature of γ.

Next it follows that $|\gamma(\psi)|^2 = h(\psi)^2 + h'(\psi)^2$. Hence $\boxed{h'(\psi)^2 \le \left(\frac{D}{2}\right)^2 \le \frac{1}{\beta^2}.}$ Altogether we have

$$|A| \le \frac{5}{\beta^3}.$$

As a result,

$$\mu(\Delta)\frac{5}{\beta^3} \ge \frac{125\pi^2}{32}d^2(h^2, U) \Rightarrow \mu(\Delta) \ge \frac{25\pi^2}{32}\beta^3 d^2(h^2, U),$$

which is the required inequality. $\square$

Open questions and discussion

1. An interesting question is whether one can consider a smaller region than $\mathcal{A}$ for the approach of Total integrability. For example, can one replace α with the invariant curve of period $8, 16$, etc..?

2. Is it possible to relax the central-symmetry restriction.

3. Recall also the old open problem on coexistence of caustics with different rotation numbers (cf. V. Kaloshin, C. E. Koudjinan).

4. It would be interesting to establish analogous results for other billiard models that lead to twist maps of the cylinder, in particular, for Outer billiards, and the recently introduced Wire billiards [B-M-T, *Adv.Math*, 368(2020)]. In this perspective together with Daniel Tsodikovich we studied rigidity and flexibility of billiard tables having symmetry of order $k \geq 3$ and an invariant curve of k-periodic orbits, *IMRN* 2022.

5. In the light of the main result of the paper, one would like to reconsider rigidity for continuous-time systems, like geodesic flows and Hamiltonian systems with a potential.

References

1. M. Bialy, A. Mironov. Angular billiard and algebraic Birkhoff conjecture. Adv. Math. 313(2017), 102–126.
2. M. Bialy. Gutkin billiard tables in higher dimensions and rigidity. Nonlinearity 31(2018), 2281–2293.
3. M.Bialy, S. Tabachnikov. Dan Reznik's identities and more. arXiv:2001.08469
4. M.Bialy, A.E.Mironov. The Birkhoff-Poritsky conjecture for centrally-symmetric billiard tables. *Ann. Math.* **2022**, *196*, 389–413.
5. M.Bialy. Effective bounds in E. Hopf rigidity for billiards and geodesic flows. Comment. Math. Helv. 90 (2015), no. 1, 139–153.
6. M.Bialy, D.Tsodikovich. Locally maximizing orbits for the non-standard generating function of convex billiards and applications. Nonlinearity 2023.
7. M.Bialy. Mather $\beta-$function for ellipses and rigidity. Entropy 24 (2022), no. 11
8. M.Bialy, D.Tsodikovich. Locally Maximizing orbits for multi-dimensional Twist maps and Birkhoff billiards. Preprint.

Thank you