Kähler-type embeddings of balls into symplectic manifolds

Michael Entov (Technion)

Joint work with

M. Verbitsky (IMPA, Rio de Janeiro & HSE, Moscow)

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(\(M^{2n}, \omega\)) – a closed symplectic manifold.

\(B^{2n}(r) \subset \mathbb{R}^{2n}\) – the closed ball of radius \(r\) centered at 0.

Q.: When can two symplectic embeddings \(\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M^{2n}, \omega)\) be mapped into each other by \(\text{Symp}_0(M, \omega)\) (\(\Leftrightarrow\) by \(\text{Ham}(M, \omega)\))? By \(\text{Symp}(M, \omega)\)? By intermediate subgroups of \(\text{Symp}(M, \omega)\)?

Thm (McDuff, 1997; previous partial results by McDuff, Lalonde, Biran): If \(\dim_{\mathbb{R}} M = 4\) and \((M, \omega)\) has “enough non-trivial Gromov-Witten invariants” (e.g. if \((M, \omega)\) is a rational or ruled surface), then for any \(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\) any two symplectic embeddings \(\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M^{2n}, \omega)\) (if they exist!) can be mapped into each other by \(\text{Symp}_0(M, \omega)\) (\(\Leftrightarrow\) the space of such sympl. emb. is path-connected).

Q.: What can be said about other (in part., higher-dim.) \((M, \omega)\)?

Will consider a more restrictive class of symplectic embeddings that can be studied using the tools of complex geometry.
Kähler structures

$M$ – closed connected oriented manifold, $\dim_{\mathbb{R}} M = 2n$.

A Kähler structure on a manifold $M$ is a pair $(\omega, J)$, where

1. $\omega$ is a symplectic form on $M$,
2. $J$ is an (integrable) complex structure on $M$ compatible with $\omega$.

A symplectic form, or a complex structure, on $M$ is said to be of Kähler type, if it appears in some Kähler structure.

Further on: $\omega$ is a fixed Kähler-type symplectic form on $M$; all symplectic forms and complex structures on $M$ are assumed to be compatible with the orientation.

$\text{Symp}_H(M, \omega)$ – the symplectom-s of $(M, \omega)$ acting trivially on $H_*(M)$.

$\mathcal{C}(M)$ – the space of (Kähler-type) complex structures on $M$.

$\mathcal{C}(M, \omega) \subset \mathcal{C}(M)$ – the space of (Kähler-type) complex structures on $M$ compatible with $\omega$. 
Teichmüller space

$\text{Teich}(M) := \mathcal{C}(M)/\text{Diff}_0(M)$ – Teichmüller space (of Kähler-type complex structures on $M$),

$pr : \mathcal{C}(M) \to \text{Teich}(M)$ – the natural projection.

$[I] := pr(I)$ – the $\text{Diff}_0(M)$-orbit of $I$. 
Kähler-type embeddings (I)

**Definition**

Let $f : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ be a symplectic embedding. First, fix a complex structure $J \in \mathcal{C}(M, \omega)$. Then $f$ is called **Kähler** (w.r.t. the Kähler structure $(\omega, J)$) if it is both symplectic (w.r.t. $\omega$) and holomorphic (w.r.t. $J$). For such an $f$, the Kähler-metric $\omega(\cdot, J\cdot) + \sqrt{-1}\omega(\cdot, \cdot)$ is flat on the image of $f$.

Now assume that no $J \in \mathcal{C}(M, \omega)$ is fixed in advance. $f$ is called **Kähler-type** if, in addition to being symplectic w.r.t. $\omega$, it is also holomorphic with respect to some *(not a priori fixed)* complex structure $J \in \mathcal{C}(M, \omega)$.

If this $J$ can be chosen to lie in the $\text{Diff}_0(M)$-orbit $[I]$ of a complex structure $I \in \mathcal{C}(M, \omega)$, we say that $f$ is of $[I]$-**Kähler type**.

More generally, if such a $J$ can be picked from a certain subset (e.g. a connected component) of $\mathcal{C}(M)$, we say that $f$ **favors** that subset.
Remarks:

1. \( f \) is of \textbf{Kähler type} \( \iff \) \( f \) is of \([I]-\text{Kähler type}\) for some \( I \in \mathcal{C}(M,\omega) \).

2. \( \exists \ [I]-\text{Kähler-type embedding into } (M,\omega) \iff \exists \text{ Kähler embedding into } (M,\omega') \) for some Kähler form \( \omega' \) on \( (M,I) \), s.t. \([\omega] = [\omega']\).

For a fixed \( I \), the existence of \([I]-\text{Kähler-type embeddings}\) was previously studied from this angle by Eckl, Witt Nystrom, Fleming, Luef-Wang, Trusiani.

3. In principle, a Kähler-type embedding may be of \([I]-\text{Kähler type}\) for different non-isotopic \( I \in \mathcal{C}(M,\omega) \) and may favor several different connected components of \( \mathcal{C}(M,\omega) \) or of \( \mathcal{C}(M) \).

4. There do exist symplectic embeddings that are \textit{not} Kähler-type (will see below).
Kähler-type embeddings (III)

Kähler-type embeddings $f, f' : \bigcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ can be connected by a smooth path in the space of the Kähler-type embeddings

\[ \iff ( \text{isotopy extension}) \]

\[ f, f' \text{ lie in the same } \text{Symp}_0(M, \omega)-\text{orbit} \]

\[ f, f' \text{ are holomorphic with respect to complex structures compatible with } \omega \text{ and lying in the same } \text{Symp}_0(M, \omega)-\text{orbit, hence in the same path-connected component of } C(M, \omega), \text{ hence in the same connected component of } C(M) \]

\[ f, f' \text{ favor the same path-connected component of } C(M, \omega) \text{ (and the same connected component of } C(M) \text{)} \]

In general – very little info on Symp$_0(M, \omega)$-orbits in $C(M, \omega)$...

However, in some cases – better info on Diff$_0(M)$-orbits of c.s. in $C(M, \omega)$, leading to results on Symp($M, \omega) \cap$ Diff$_0(M)$-action on Kähler-type emb. (Note that $C(M, \omega)$ is not Diff$_0(M)$-invariant!)
\( \hat{M}^k := \text{the space of } k\text{-tuples of pairwise distinct points of } M, \)

\( \mathbf{x} := (x_1, \ldots, x_k) \in \hat{M}^k. \)

\( I \) – a complex structure on \( M \).

\( \widetilde{M}_{I,\mathbf{x}} \) – the complex blow-up of \((M, I)\) at \( x_1, \ldots, x_k \).

\( \widetilde{I} \) – the lift of \( I \) to \( \widetilde{M}_{I,\mathbf{x}} \).

\( \Pi : \widetilde{M}_{I,\mathbf{x}} \to M \) – the natural projection.

\( e_1, \ldots, e_k \in H^2(\widetilde{M}_{I,\mathbf{x}}; \mathbb{R}) \) – the cohomology classes Poincaré-dual to the homology classes of the exceptional divisors.

**Definition**

For \( r_1, \ldots, r_k > 0, \ \mathbf{r} = (r_1, \ldots, r_k) \), define \( K(\mathbf{r}) \subset C(M, \omega) \) as

\[
K(\mathbf{r}) := \left\{ I \in C(M, \omega) \mid \exists \mathbf{x} \in \hat{M}^k \ \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\widetilde{M}_{I,\mathbf{x}}; \mathbb{R}) \text{ is Kähler w.r.t. } \widetilde{I} \right\}.
\]
Existence – main result

Theorem

Let $r_1, \ldots, r_k > 0$, $\mathbf{r} = (r_1, \ldots, r_k)$. The following are equivalent:

- $\exists$ a Kähler-type embedding $\bigcup_{i=1}^{k} B^{2n}(r_i) \rightarrow (M, \omega)$.
- $K(\mathbf{r}) \neq \emptyset$ (i.e., $\exists I \in C(M, \omega)$ and $x \in \hat{M}^k$ s.t. $\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\tilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\tilde{I}$).

More precisely, if $I \in C(M, \omega)$, then the following are equivalent:

- $\exists x \in \hat{M}^k$ s.t. $\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\tilde{M}_{I,x}; \mathbb{R})$ is Kähler w.r.t. $\tilde{I}$.
- $\exists [I]$-Kähler-type embedding $\bigcup_{i=1}^{k} B^{2n}(r_i) \rightarrow (M, \omega)$.

Remarks:

1. The proof is a modification of the proof of a similar result for symplectic embeddings of balls (McDuff-Polterovich, 1994).

2. A similar existence result holds for $[I]$-Kähler-type embeddings into $(M \setminus \Sigma, \omega)$ for a proper complex submanifold $\Sigma \subset (M, I)$. 
Theorem

Let $C_0$ be a connected component of $C(M)$. Assume that $\text{pr} \left( K(\mathbb{R}) \cap C_0 \right) \subset \text{Teich}(M)$ is connected.

Then any two $K$-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ favoring $C_0$ lie in the same $\text{Symp}(M, \omega) \cap \text{Diff}_0(M)$-orbit. In particular, both embeddings are of $[I]$-Kähler type for the same $I$.

If, in addition, $\text{Symp}_H(M)$ acts transitively on the set of connected components of $C(M)$ intersecting $C(M, \omega)$, then any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ lie in the same $\text{Symp}_H(M, \omega)$-orbit.
Remark:

Assume \( \dim_{\mathbb{R}} M = 4 \) and \((M, \omega)\) has “enough non-trivial Gromov-Witten invariants” (e.g. if \((M, \omega)\) is a rational or ruled surface).

Then, for any \( \bigsqcup_{i=1}^{k} B^4(r_i) \), any two Kähler-type embeddings \( \bigsqcup_{i=1}^{k} B^4(r_i) \to (M^4, \omega) \) (if they exist!) can be mapped into each other by \( \text{Symp}_0(M, \omega) \) (since, by McDuff’s thm., the same is true for any two symplectic embeddings).

Consequently, as long as there exists a Kähler-type embedding \( \bigsqcup_{i=1}^{k} B^4(r_i) \to (M^4, \omega) \), all symplectic embeddings \( \bigsqcup_{i=1}^{k} B^4(r_i) \to (M^4, \omega) \) are of Kähler-type – because the set of Kähler-type embeddings is \( \text{Symp}_0(M, \omega) \)-invariant.
Theorem

In the following cases we have necessary and sufficient conditions for the existence of Kähler-type embeddings \( \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS}) \):

A. \( k = l^n, r_1 = \ldots = r_k =: r \): \( \text{Vol} (\bigsqcup_{i=1}^{k} B^{2n}(r)) < \text{Vol} (\mathbb{C}P^n, \omega_{FS}) \).

B. \( n = 2, 1 \leq k \leq 8 \): \( \text{Vol} (\bigsqcup_{i=1}^{k} B^{4}(r_i)) < \text{Vol} (\mathbb{C}P^2, \omega_{FS}) \) & additional explicit quadratic inequalities on \( r_1, \ldots, r_k > 0 \) (coming from the description of the Kähler cone of the blow-up of \( \mathbb{C}P^2 \) at \( k \) generic points).

In both cases for any complex structure \( I \) on \( \mathbb{C}P^n \) compatible with \( \omega_{FS} \) (and, in particular, for the standard complex structure \( I_{st} \)), there exists an \([I]-\text{Kähler-type embedding} \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS}) \).
Corollary

Let $I_{st}$ be the standard complex structure on $\mathbb{C}P^n$. Assume that $\text{Vol}(B^{2n}(r)) < \text{Vol}(\mathbb{C}P^n, \omega_{FS})$.

Then there exists a Kähler form $\omega$ on $(\mathbb{C}P^n, I_{st})$ isotopic to $\omega_{FS}$ and such that the Kähler manifold $(\mathbb{C}P^n, I_{st}, \omega)$ admits a Kähler (that is, both holomorphic and symplectic) embedding of $B^{2n}(r)$ with the standard flat Kähler metric on it.

For $n = 2$ this was previously proved by Eckl (2017).

Remark:

For any $\bigsqcup_{i=1}^{k} B^4(r_i)$, any Kähler-type embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$ (if it exists!) is, in fact, of $[I_{st}]$-Kähler-type.

If $k = l^2$ and $r_1 = \ldots = r_k$ or if $1 \leq k \leq 8$, then any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$ is of Kähler-type – in fact, of $[I_{st}]$-Kähler-type.
Sample applications: $\mathbb{C}P^n$ (III)

**Theorem**

For any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$, any two Kähler-type embeddings

$$\bigsqcup_{i=1}^k B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS})$$

can be mapped into each other by $\text{Symp}(\mathbb{C}P^n, \omega_{FS}) = \text{Symp}_H(\mathbb{C}P^n, \omega_{FS})$. They can be mapped into each other by $\text{Symp}(\mathbb{C}P^n, \omega_{FS}) \cap \text{Diff}_0(\mathbb{C}P^n)$ if and only if they favor the same connected component of $\mathcal{C}(\mathbb{C}P^n)$.

**Remarks:**

1. For $n = 2$ the group $\text{Symp}(\mathbb{C}P^2, \omega_{FS})$ is connected (Gromov) and thus for any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$ the space of Kähler-type embeddings $\bigsqcup_{i=1}^k B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$ is path-connected (as also follows from McDuff’s thm. about symplectic embeddings in dim. 4).

2. For $n = 3$ (Kreck-Su) and $n = 4$ (Brumfiel) the space $\mathcal{C}(\mathbb{C}P^n)$ has more than one connected component. Unknown for other $n$.

3. For $n > 2$ it is unknown if $\text{Symp}(\mathbb{C}P^n, \omega_{FS})$ is connected or lies in $\text{Diff}_0(\mathbb{C}P^n)$.
Assume $M$ is either $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ or a smooth manifold underlying a K3 surface; $\omega$ is a Kähler-type symplectic form on $M$.

**Remark:** The Kähler-type symplectic/complex structures on $\mathbb{T}^{2n}$ (compatible with the standard orientation) are exactly the ones that can be mapped by a diffeomorphism of $\mathbb{T}^{2n}$ to a linear symplectic/complex structure. It is unknown if there exist non-Kähler-type symplectic forms on $\mathbb{T}^{2n}$, $n > 1$, or on K3 surfaces.

**Definition**

The form $\omega$ is called **rational** if $[\omega] \in H^2(M; \mathbb{R})$ is proportional to a rational homology class, and **irrational** otherwise.
Theorem

Assume $M$ is either $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ or a smooth manifold underlying a K3 surface. Let $\omega$ be a Kähler-type symplectic form on $M$. Assume that either $\omega$ is irrational or $M = \mathbb{T}^2$. Then:

$$\text{Vol} \left( \bigcup_{i=1}^{k} B^{2n}(r_i) \right) < \text{Vol} (M, \omega)$$

$$\exists \text{ a Kähler-type embedding } \bigcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$$

Remark:

For $M = \mathbb{T}^{2n}$ and one ellipsoid – and, in particular, for one ball – this was previously proved by Luef and Wang (2021) using a similar method. Their work relates the problem for $M = \mathbb{T}^{2n}$ to Gabor frames (an important notion in signal processing).
Theorem

Let $\omega$ be a Kähler-type symplectic form on $M$. Assume that either $\omega$ is irrational or $M = \mathbb{T}^2$.

Then for any $k \in \mathbb{Z}_{>0}$ and any $r_1, \ldots, r_k > 0$ any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ can be mapped into each other by $\text{Symp}_H(M, \omega)$. They can be mapped into each other by $\text{Symp}(M, \omega) \cap \text{Diff}_0(M)$ if and only if they favor the same connected component of $\mathcal{C}(M)$.

Remarks:

1. It is unknown whether $\text{Symp}_H(\mathbb{T}^{2n}, \omega) = \text{Symp}_0(\mathbb{T}^{2n}, \omega)$ for any Kähler-type symplectic form $\omega$ on $\mathbb{T}^{2n}$, $n > 1$.

2. In the K3 case, for at least some irrational $\omega$:
   $\text{Symp}_0(M, \omega) \subset \subset \text{Symp}_H(M, \omega)$ (Sheridan-Smith, 2020),
   $\text{Symp}_0(M, \omega) \subset \subset \text{Symp}(M, \omega) \cap \text{Diff}_0(M)$ (Seidel, 2000; Smirnov, 2022).
Remark: If the Kähler-type form $\omega$ on $\mathbb{T}^{2n}$ is rational, then there may be obstructions to the existence of Kähler-type embeddings of balls into $(M, \omega)$ that are independent of the symplectic volume – for instance, obstructions coming from Seshadri constants.

Example: Let $M = \mathbb{T}^{4}$, $\omega = dp_{1} \wedge dq_{1} + dp_{2} \wedge dq_{2}$, $\text{Vol}(\mathbb{T}^{4}, \omega) = 2$. For any complex structure $I$ on $\mathbb{T}^{4}$ compatible with $\omega$ one can biholomorphically identify $(\mathbb{T}^{4}, I)$ with a principally polarized abelian variety. A universal upper bound on the Seshadri constants for all such varieties (Steffens, 1998) yields that if $(4/3)^{2} < \text{Vol}(B^{4}(r)) < 2$, then there are no Kähler-type embeddings $B^{4}(r) \rightarrow (\mathbb{T}^{4}, \omega)$. However, for such $r$ there do exist symplectic embeddings $B^{4}(r) \rightarrow (\mathbb{T}^{4}, \omega)$ (Latschev-McDuff-Schlenk, 2013; E.-Verbitsky, 2016).
Definition

Assume:

$M^{2n}$ is a closed manifold, $\omega$ is a Kähler-type symplectic form on $M$;
$\mathbb{W}$ is a disjoint union of compact domains with boundary in $\mathbb{R}^{2n}$.

For $\varepsilon > 0$, a symplectic embedding $\mathbb{W} \to (M, \omega)$ is called $\varepsilon$-tame if it is holomorphic w.r.t some (not a priori fixed!) complex structure $I$ on $M$ which is “$\varepsilon$-almost compatible with $\omega$” – i.e., such that

1. $I$ is tamed by $\omega$,
2. the cohomology class $[\omega]^{1,1}_I$ is Kähler,
3. $\left| \langle \left( [\omega]^{2,0}_I + [\omega]^{0,2}_I \right)^n , [M] \rangle \right| < \varepsilon$.

Remark: For symplectic embeddings $\mathbb{W} \to (M, \omega)$:
Kähler-type $\implies$ $\varepsilon$-tame for every $\varepsilon > 0$. 
Theorem

Assume that $M$ is either $\mathbb{T}^{2n}$ or a smooth manifold underlying a K3 surface and $\mathbb{W} := \bigcup_{i=1}^{k} W_i$ is a disjoint union of compact domains with boundary specified in the next slide.

Then for any $\varepsilon > 0$ there exists a $\text{Diff}^+(M)$-invariant open dense set $\Theta(\mathbb{W}, \varepsilon)$ of Kähler-type symplectic forms on $M$ (depending on $\mathbb{W}$ and $\varepsilon$ and containing, in particular, all irrational Kähler-type symplectic forms on $M$), so that for each $\omega \in \Theta(\mathbb{W}, \varepsilon)$, the only obstruction to the existence of $\varepsilon$-tame symplectic embeddings $\mathbb{W} \to (M, \omega)$ is the symplectic volume.

This holds (at least) if $\mathbb{W}$ is either of the following...
... This holds (at least) if $W$ is either of the following:

- a disjoint union of $k$ (possibly different) $2n$-dimensional balls,
- a disjoint union of $k$ identical copies of a parallelepiped $P(e_1,\ldots,e_{2n}) := \left\{ \sum_{j=1}^{2n} s_j e_j, 0 \leq s_j \leq 1, j = 1,\ldots,2n \right\}$, spanned by a basis $e_1,\ldots,e_{2n}$ of $\mathbb{R}^{2n}$.

If $M = \mathbb{T}^{2n}$, we also allow $W$ to be a disjoint union of $k$ identical copies of a $2n$-dim. polydisk $B^{2n_1}(r_1) \times \ldots \times B^{2n_l}(r_l)$, $n_1 + \ldots + n_l = n$.

**Remark:** If $M = \mathbb{T}^{2n}$, then for all $W$ above and all $\varepsilon > 0$, the open dense set $\Theta(W,\varepsilon)$ appearing in the theorem contains a $\text{Diff}^+(\mathbb{T}^{2n})$-orbit of an irrational K.-type form, so that for each form $\omega$ in the orbit, the only obstruction to the existence of Kähler-type symplectic embeddings $W \to (M,\omega)$ is the symplectic volume. (The orbit is dense in the set of all Kähler-type symplectic forms of a fixed volume on $\mathbb{T}^{2n}$).
THANK YOU!