The Toda lattice, billiards and symplectic geometry

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Symplectic topology

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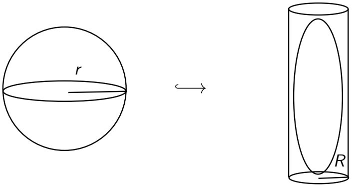

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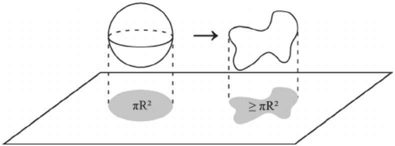

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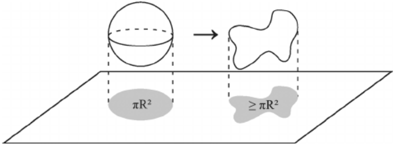

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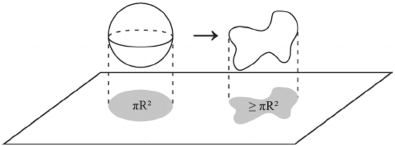

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The existence of a normalized symplectic capacity is equivalent to Gromov's nonsqueezing theorem.

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- First embedded contact homology capacity c_1^{ECH} (Hutchings 2011) - only in dimension 4.

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For any compact set X,

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If $X \subset \mathbb{R}^{2n}$ is a compact and convex set and c is a normalized symplectic capacity, then

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Moreover equality holds if, and only if, $\mathrm{int}(X)$ is symplectomorphic to a ball.

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Moreover equality holds iff $\mathrm{int}(X) \cong B^{2n}$.

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Equality is attained for the hypercube $K = I^n$.

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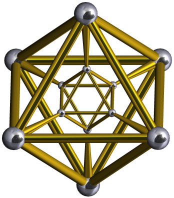

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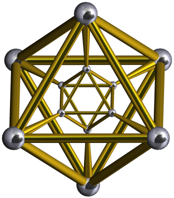

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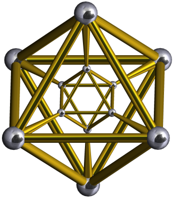

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$M(K)$ is minimized precisely by the Hanner polytopes.

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$$\text{Viterbo} \Rightarrow \text{Weak Viterbo} \Rightarrow \text{Mahler}$$

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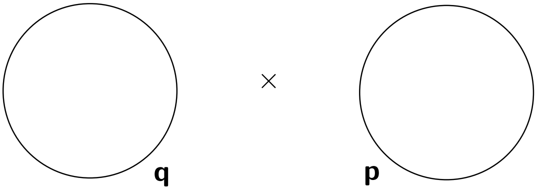

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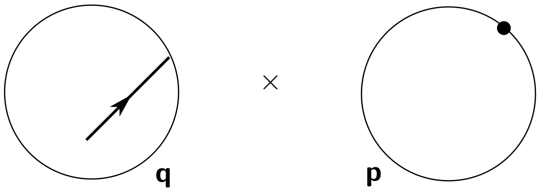

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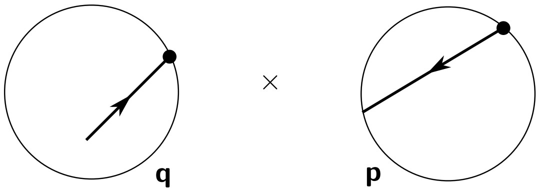

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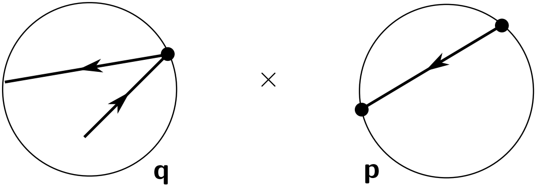

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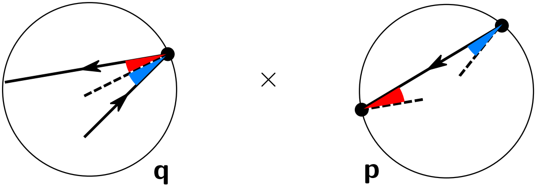

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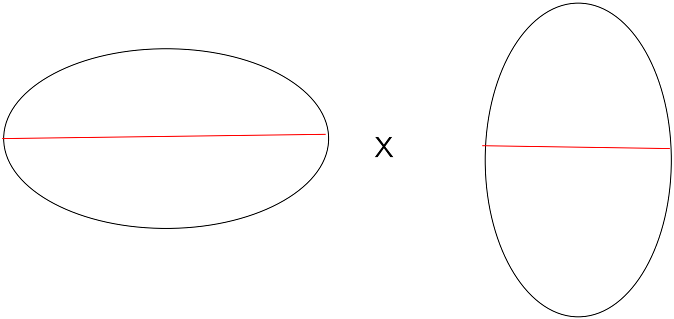

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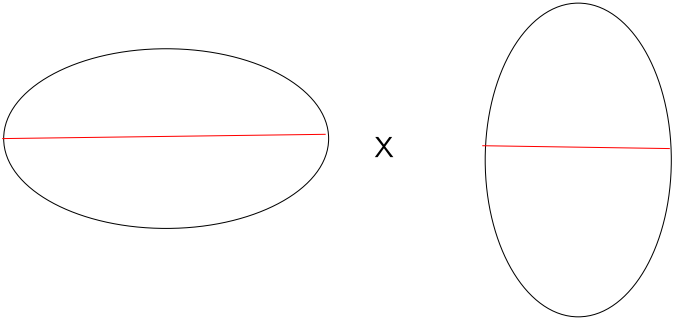

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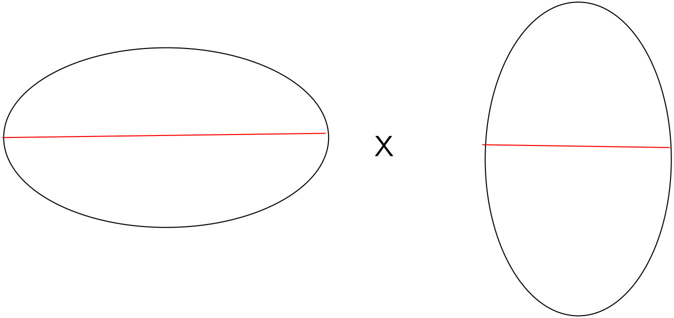

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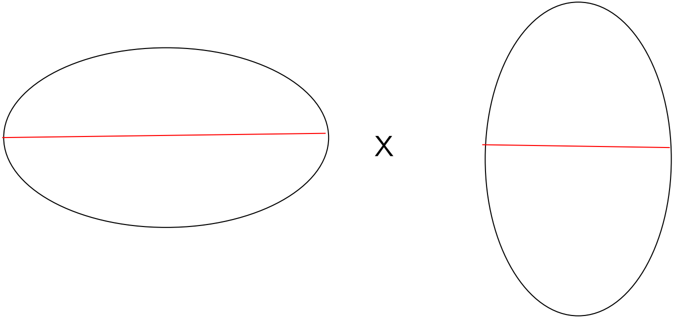

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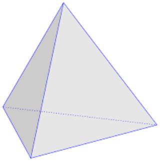

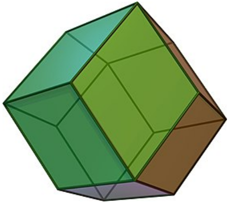

Toric domains

Definition

A toric domain $\mathbb{X}_\Omega \subset \mathbb{C}^n$ is a set of the form $\mathbb{X}_\Omega = \mu^{-1}(\Omega)$, where $\Omega \subset [0, \infty)^n$ is an open set and

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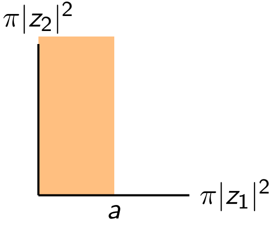

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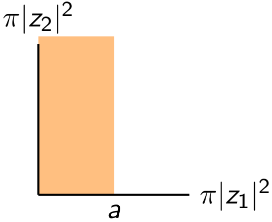

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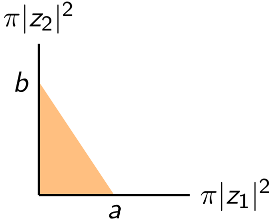

$E(a,b) := \left\{(z_1,z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \le 1\right\}$

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If A is symmetric, then A is determined by $\rho(A)$.

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Theorem (Ostrover–R.–Sepe, 2023)

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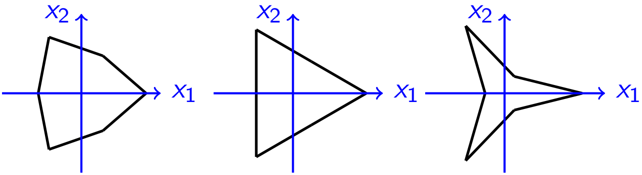

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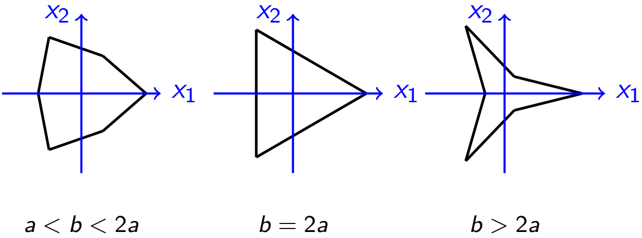

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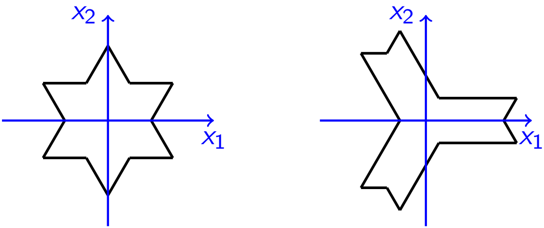

Figure: *The domain A for which $\mathbb{X}_{\rho(A)}$ is $P(1, 1)$ and $P(1, 3)$*

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$$\begin{cases} 0, \text{ if } q_i - q_{i+1} < 1, \text{ for all } i = 1, \ldots, n, \\ \infty, \text{ if } q_i - q_{i+1} \geq 1, \text{ for some } i = 1, \ldots, n. \end{cases}$$

The flow of X_{H_c} converges to the billiard flow in

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$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + \sum_{i=1}^{n+1} e^{q_i - q_{i+1}}.$$

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Lax pair formulation

There exists a Lax pair (L, B) such that the Hamiltonian system above is equivalent to $\dot{L} = [L, B]$,

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_{n+1} \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & 0 & 0 & \dots & b_{n+1} \end{pmatrix}.$$

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Theorem (Hénon 1973)

The Toda lattice is completely integrable.

The Arnold-Liouville theorem

Fix (M^{2n}, ω) and let $F = (H_1, \ldots, H_n) : M \to \mathbb{R}^n$ such that $\{H_i, H_j\} = 0$ for all i, j.

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$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \mathbb{X}_\Omega \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \mu} \\ F(U) & \xrightarrow{\phi} & \Omega \end{array}$$

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- The map ϕ can be obtained by action coordinates:

$$\phi(c) = \left(\oint_{\gamma_1^c} \lambda, \ldots, \oint_{\gamma_n^c} \lambda \right).$$

Spectral theory for the Toda lattice

Difference equation related to the eigenvalue problem of L:

$$a_{k-1}y_{k-1}(\lambda) + b_k y_k(\lambda) + a_k y_{k+1}(\lambda) = \lambda y_k(\lambda).$$

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Theorem (Flaschka–McLaughlin, van Moerbeke, Moser)

The map $\left\{(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n+2} \mid \sum_i p_i = \sum_i q_i = 0\right\} \to \mathbb{R}^{2n}$ *defined by* $(q, p) \mapsto (f_1, \ldots, f_n, \mu_1, \ldots, \mu_n)$ *is a symplectomorphism.*

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Let $D(\lambda)$ be the discriminant of the difference equation in the previous slide.

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The action coordinates $\phi = (J_1, \ldots, J_n)$ are given by

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We can define action coordinates for the deformation of the Toda lattice:

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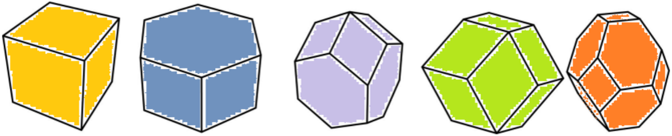

Figure: The Fedorov polyhedra

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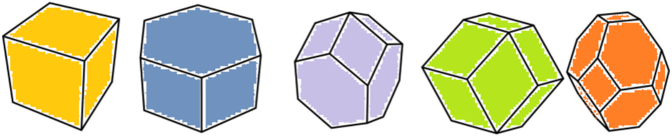

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Question 2

Do other root systems B_n, C_n, D_n, G_2, etc, give rise to interesting symplectomorphisms?