Local exotic tori

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Outline:

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§0. Overview.

\((X^{2n}, \omega) \to L^n\) Lagrangian \(\Rightarrow w \mid_{T_L} = 0\)

We care about Lagrangian tori.

Ambitions goal: Classify them up to Symp or Ham.

e.g. 1) \(X = \mathbb{R}^2 = \mathbb{C}\)

Every Lagrangian torus is Ham, isotopic to \(S^1(a)\) for some \(a > 0\).

2) \(X = \mathbb{R}^4 = \mathbb{C}^2\)

\(T(a_1, a_2) = S^1(a_1) \times S^1(a_2)\) "product torus"
Question: \( L \subset C^2 \) Lagrangian torus
\( 3a_1, a_2 > 0 \) s.th.
\( L \cong T(a_1, a_2) \) ?

No. Chekanov ('96)
\( T_{ch}(a) \subset C^2 \)
uses capacities and versal deformations to distinguish tori.

Eliashberg - Polterovich ('96)
different construction
uses count of J-holomorphic disks with bdry on \( L \) to distinguish tori.

Def: Lagrangian tori \( L \subset C^n \)
\( L \not\cong T(a_1, \ldots, a_n) \)
are called exotic.

The classification of Lagrangian tori in \( C^2 \) is still open.

(prospect made by Dimidroglo - Pizzell '13)
Search for exotic tori:

* Chekanov - Schlenk (10):
  "twist tori" in $S^2 \times \ldots \times S^2$ and $\mathbb{C}P^n$
  (uses versal deformations)

* Auroux (15):
  infinitely many monotone tori in $\mathbb{C}^3$
  (uses cont of $J$-hol. disks)

* Vianna (16, 17):
  infinitely many monotone tori in $\mathbb{C}P^2$
  & other Del Pezzo surfaces
  (uses cont of $J$-hol. disks)

Most exotic tori that were studied are monotone, i.e., Maslov and
area classes are pos. proportional.

Exception: Shelukhin - Tonkonog - Vianna (19)

F000: $S^2 \times S^2$
non-monotone exotic tori

(can also be detected by versal deformations)
§ 1. Results.

(All results hold in $\dim \geq 6$, but are stated for $\dim = 6$.)

For every $k \in \{-1, 0, 1, 2, \ldots, 3\}$, define

$$X_k(a_1, a_2) \subset \mathbb{C}^3, \quad a_1, a_2 > 0$$

two parameter family of log. tori.

with $X_k(a_1, a_2)$, $X_k'(a_1, a_2)$ have same area and Maslov class, but:

**Thm.** (B. ’23)

$$X_k(a_1, a_2) \neq X_k'(a_1, a_2) \text{ for } k \neq k'.$$

**Remarks:** 1) One parameter sub-family of monotone tori

$$Y_k(0, a_2) \subset \mathbb{C}^3$$

2) ReproducesFantoux’s result different construction & invts.
Every ball $B^6(R)$ contains $X_k(a_1,a_2)$ for $a_1,a_2$ small enough.

Let $\varphi : B^6(R) \rightarrow (X^6,\omega)$ DARKER chart.

**Question:** Are $\varphi(X_k)$, $\varphi(X_{k'})$ still distinct?

**Thm.** (B., '23)

Let $(X^6,\omega)$ be tame and $\varphi$ DARKER chart. Then $\exists \varepsilon > 0$ s.t.

$$\varphi(X_k) \neq \varphi(X_{k'}) \quad k \neq k'$$

for all $X_k, X_{k'} \subset B^6(\varepsilon) \subset B^6(R)$.

**Local exotic horn**

**Marked observation:** The displacement energy germ (induced by versal def.) doesn't change under the embedding $\varphi$ (provided $\varepsilon > 0$ small enough).
§ 2. Construction.

Inspired by the Eliashberg–Polterovich construction of the Chekanov torus.
(see also Chekanov–Schlenk, Auroux ...)

Review: $\mathbb{R} \mathbb{C} = \{ (z_1, z_2) \}, w_0$

\[ H = \pi(1z_1^2 - 1z_2^2) \]
generates $S^1$-action. Perform symplectic reduction.

$H^{-1}(c) \not\subset S^1$ free $\forall c \neq 0$

$H^{-1}(0) \not\subset S^1$ not free

not smooth $(0,0) \in \mathbb{C}^2$ fixed pt.)

$C \neq 0$:

$H^{-1}(c)/S^1 \simeq (C, w_0)$

product tori $\leftrightarrow$ circles

$\subset H^{-1}(c) \subset \mathbb{C}$

$T(a_1, a_2) \leftrightarrow$
In the toric picture of $\mathbb{C}^2$: $\mu_1 \neq \mu_2$

$$\mu : \mathbb{C}^2 \to \mathbb{R}_{>0}^2, \quad \mu(z_1, z_2) = (\sqrt{z_1^2}, \sqrt{z_2^2})$$

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H = \mu_1 - \mu_2

fibres of $\mu$

product tori.
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$RH : H = \langle \mu_1, \nu \rangle$

$\nu = (1, -1)$

This is a general recipe

rational segment $\to$ symplectic $\to$ product tori

from $\mathbb{C}^n$ to surface

cf. probes by McDuff.
Construction of $\Gamma_k(a_1,a_2) \subset \mathbb{C}^3$:

Symplectic reduction on segments $w/\text{direction}$

\[ (-k) \in \mathbb{Z}^3 \]

$\mathbb{R}^3_{\geq 0}$

Symplectic reduction by $T^2$-action.

For $k \geq 2$, get orbifold reduced spaces with one puncture:

$\Gamma_k(a_1,a_2)$

Def.: The tori $\Gamma_k$ are lifts of red curves in the red spaces.

$\mapsto$ product tori.
§ 3. Versal deformations and displacement energy.

(Chekanov ’96 see also C.-S. ’10, ’16 B. ’20)

Recall: displacement energy.

\[ e(L) = \inf \left\{ \|H\| \mid \eta^H(L) \cap L = \emptyset \right\} \]

Refer norm \( (= \infty \text{ if empty}) \)

E.g.

\[ e(S^1(a)) = a \]

\[ e(T(a_1, \ldots, a_n)) = \oplus \text{ min } a_1, \ldots, a_n \]

Thm. (Chekanov ’98)

\((X, \omega)\) tame and \( J \) almost complex str. taming \( \omega \). \( L \subseteq X \) compact Lagrangian.

Then:

\[ e(L) \geq \min \left\{ \text{min. area of } J \text{-curves in } (X, J), \right. \]

\[ \text{min area of } J \text{-disks w/ bdry on } L \} \]
Unfortunately,
\[ e(T_{Ce}(a)) = e(T_{Ca}(a)) = a \]

But \( e(-) \) behaves differently on neighborhoods of \( T_{Ce}(a) \) and \( T_{Ca}(a) \) in the space of Lagrangian tori. (Versal def.)

More precisely:
\[ L \subset (X,\omega) \text{ cpt.} \]
\[ \{ C^1\text{-small neigh. of } L \} \cong U \subset H^1(L;\mathbb{R}) \]

Ham. isotopies supp. in Weinstein chart.

Get a function
\[ H^1(L;\mathbb{R}) \ni U \overset{e(-)}{\to} \mathbb{R} \cup \{ 1+\infty \} \]

Def.: The displacement energy germ \( E_L : H^1(L;\mathbb{R}) \to \mathbb{R} \cup \{ 1+\infty \} \) is the germ of \((\star)\) at 0.
Prop: Symplectic invariance:
\[
\mathcal{E}_\phi(L) \circ \phi^* = \mathcal{E}_L
\]
\(\forall \phi \in \text{Symp}(X,\omega)\).
\(H^1(T_{\text{ce}}(a); \mathbb{R}) \cong \mathbb{R}^2\)

e.g. \[\mathcal{E}_{T_{\text{ce}}(a)}(b_1, b_2) = \alpha + \min \{b_1, b_2\}\]

The "versal deformation" of \(T_{\text{ce}}(a)\) is just \((b_1, b_2) \mapsto T(\alpha + b_1, \alpha + b_2)\).
\[\mathcal{E}_{T_{\text{ch}}(a)}(c_1, c_2) = \alpha + c_1\]

Cor.: \(T_{\text{ch}}(a)\) is exotic.

Versal deformation of \(T_{\text{ch}}(a)\):
\[
\{ U \subset H^1(T_{\text{ch}}(a); \mathbb{R}) \}
\]

\(\{ T_{\text{ch}}(b) \}_{b \in \ldots}\) hom. isotopic to product tori.
(!) It is enough to compute $\Sigma_{Tc(a)}$ on $U \setminus$ red segment, i.e. to know $e(T(a_1, a_2))$.

Why?

$\mathcal{U}^{-1}(\mathcal{I})/\mathcal{I}^1$

Deformation of $Tc(a)$

Same applies to $\mathcal{I}_k(a_1, a_2)$:

$\Sigma_{Tc(a_1, a_2)}(x, y, z) = \begin{cases} a_2 + \min \{x, x + ky, z\} & a_1 = 0 \text{ (monotone)} \\ a_2 + \min \{x, x + kz\} & \text{non-monotone case} \end{cases}$
For \( k \neq k' \) \( \mathcal{F} \in \text{GL}(3, \mathbb{Z}) \) s.t.
\[
\mathcal{F}_k \circ \mathcal{F} = \mathcal{F}_k.
\]

§4. Embeddings into tame manifolds.

Let \( \phi : B^6(R) \to X^6 \) be a compact chart.

Goal: Show that for small \( \Xi_k \)
\[
\mathcal{E} \phi(\Xi_k) = \mathcal{E} \Xi_k
\]

in \( X^6 \) in \( \mathbb{C}^3 \)
on an open dense subset.

Compute \( \epsilon(\phi(T)) \) for \( T \) near \( \Xi_k \).

Pick \( \mathcal{F} \) tame extending \( \phi + \phi_0 \) on \( \text{im} \phi \).

Take \( \epsilon < \min \left\{ \frac{\sqrt{2}}{2}, \text{area of smallest U-curve in } (X, f) \right\} \)
and take \( \Xi_k \subset B^6(\epsilon). \)
Since $\varepsilon < R/2$, $T$ can be displaced in $B^6(R)$.

$e(\varphi(T)) \leq e(T)$.

For the other inequality use Chekanov's Theorem.

1) J-holomorphic disks w/ border on $\varphi(T)$

Disks contained in $\text{im } \varphi$ are understood.

Claim: Minimal J-disk in $(X,J)$ = Minimal J-disk in $B^6(R)$ for all $T \subseteq B^6(\varepsilon)$.

Prop. (C-S, '16) If the disk leaves $B^6(R)$, then $\text{area}_s > m$. 
2) \( j \)-hol. curves in \((X,j)\) do not interfere, since
\[ \varepsilon < \min \text{ area of } j\text{-curve in } (X,j). \]

§ 5. Further directions.

1. What about dimension four?
   
   Same argument applies to distinguish \( \varphi(T_{CE}), \varphi(T_{CH}) \)

   **But:** Sometimes \( \varphi(T_{CH}) \) is not exotic in a "meaningful" way:
   
   Ex. : \( X^4 \) toric and
   \[ \varphi(T_{CH}) \cong \text{toric fibre}. \]

   Next best: Construct exotic tori in the \( \text{Ubhd of Lagrangian } S^2, \text{RP}^2, \text{pinwheels.} \)
Ongoing discussions with J. Schmitz

\[ L = \text{Lagrangian sphere } \mathbb{C} \times \mathbb{C}^2 \]

In a Weilstein chart, have

\[ T_{\alpha}, T_{\chi}, T_{P,1} \subset T^* S^2 \]

Show that they are distinct in \( x^4 \).

2. In dimension 4, \( T_{\alpha} \subset C^2 \) is a fibre of an almost

\[ \text{toric fibration.} \]

Idea: “Extend” \( S^\infty \) to fibration

on \( C^3 \) should give an idea of how to get

higher dimensional ATFs.

(The versal deformation is a local such fibration.)
Thank You!