Local exotic tori

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Outline:

§0. Exotic tori: Overview

§1. Results

§2. Construction in $\mathbb{C}^3$

§3. Versal deformations & Displacement energy.

§4. Embeddings into tame manifolds

§5. Further directions.

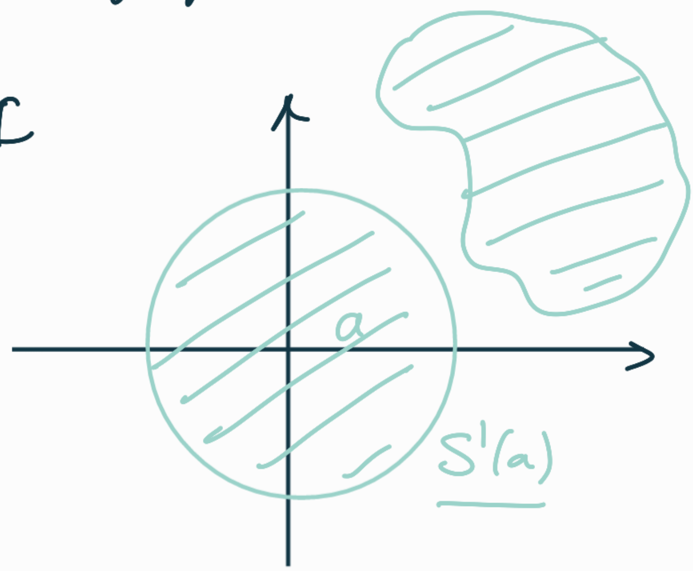
§0. Overview.

$(X^{2n}, \omega) \supset L^n$ Lagrangian $\iff \omega|_{TL} = 0$

We care about Lagrangian tori.

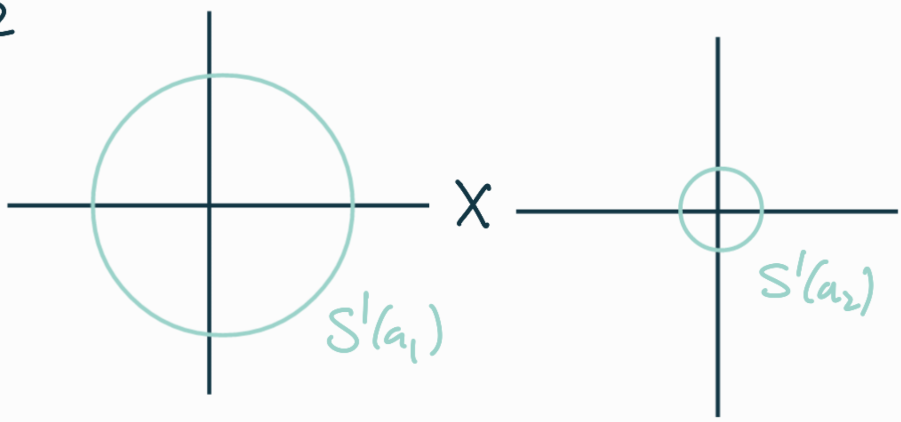
<u>Ambitious goal</u>: Classify them up to Symp or Ham.

e.g. 1) $X = \mathbb{R}^2 = \mathbb{C}$



Every Lagrangian torus is Ham. isotopic to $S^1(a)$ for some $a > 0$.

2) $X = \mathbb{R}^4 = \mathbb{C}^2$



$T(a_1, a_2) = S^1(a_1) \times S^1(a_2)$ "product torus"

Question: $L \subset \mathbb{C}^2$ Lagrangian torus $\exists a_1, a_2 > 0$ s.th.

$$L \underset{\text{Ham. diff.}}{\cong} T(a_1, a_2) \quad ?$$

No. Chekanov ('96) $T_{Ch}(a) \subset \mathbb{C}^2$ uses capacities and versal deformations to distinguish tori.

Eliashberg-Polterovich ('96) different construction uses count of J-holom. disks with bdry on L to distinguish tori.

Def: Lagrangian tori $L \subset \mathbb{C}^n$ w/

$$L \not\cong T(a_1 \ldots a_n)$$

are called exotic.

$$\boxed{\text{The classification of Lagrangian tori in } \mathbb{C}^2 \text{ is still open.}}$$

(progress made by Dimitroglou-Rizell '19)

Search for exotic tori:

*) Chekanov – Schlenk ('10):
"twist tori" in $S^2 \times \ldots \times S^2$, $\mathbb{CP}^n$.
(uses versal deformations)

*) Auroux ('15):
infinitely many monotone tori in $\mathbb{C}^3$
(uses count of J-hol. disks)

*) Vianna ('16, '17):
infinitely many monotone tori in $\mathbb{CP}^2$
& other Del Pezzo surfaces
(uses count of J-hol. disks)

Most exotic tori that were studied are monotone, i.e. Maslov and area classes are pos. proportional.

Exception: Shelukhin – Tonkonog – Vianna ('19)
non-monotone exotic tori in $\mathbb{CP}^2$.
(can also be detected by versal deformations)

FOOO: $S^2 \times S^2$ non-mon. exotic tori.

§1. Results.

(All results hold in dim ≥ 6, but are stated for dim $= 6$.)

For every $k \in \{\underbrace{-1, 0, 1, 2, \ldots}\}$, define

$$\Upsilon_k(a_1, a_2) \subset \mathbb{C}^3, \quad \boxed{a_1, a_2 > 0}$$

two parameter family of Lag. tori.

with $\Upsilon_k(a_1, a_2)$, $\Upsilon_{k'}(a_1, a_2)$ have same area and Maslov class, but:

Thm.: (B. '23)

$$\Upsilon_k(a_1, a_2) \not\cong \Upsilon_{k'}(a_1', a_2') \text{ for } k \neq k'.$$

Remarks: 1) One parameter sub-family of monotone tori

$$\Upsilon_k(0, a_2) \subset \mathbb{C}^3$$

2) Reproduces Auroux's result different construction & invts.

Every ball $B^6(R)$ contains $\Upsilon_k(a_1,a_2)$ for a_1, a_2 small enough.

Let $\varphi : B^6(R) \hookrightarrow (X^6, \omega)$ Darboux chart.

tame

<u>Question:</u> Are $\varphi(\Upsilon_k)$, $\varphi(\Upsilon_{k'})$ still distinct?

<u>Thm.:</u> (B. '23)

Let (X^6, ω) be tame and φ Darboux chart. Then $\exists\, \varepsilon > 0$ s.th.

$$\varphi(\Upsilon_k) \not\cong \varphi(\Upsilon_{k'}) \qquad k \neq k'$$

for all $\boxed{\Upsilon_k, \Upsilon_{k'} \subset B^6(\varepsilon)} \subset B^6(R)$.

"local exotic tori"

<u>main observation:</u> The displacement energy germ (induced by versal def.) doesn't change under the embedding φ (provided $\varepsilon > 0$ small enough)

§2. Construction.

Inspired by the Eliashberg - Polterovich construction of the Chekanov torus.
(see also Chekanov - Schlenk ('10), Auroux ...)

Review: in $(\mathbb{C}^2 = \{(z_1, z_2)\}, \omega_0)$

$$H = \pi(|z_1|^2 - |z_2|^2)$$

generates S^1-action. Perform symplectic reduction.

$$H^{-1}(c) \circlearrowleft S^1 \text{ free} \quad \forall c \neq 0$$

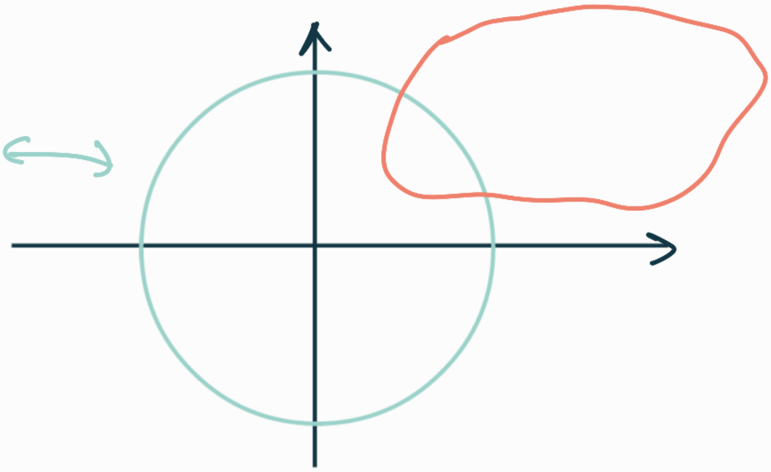
$H^{-1}(0) \circlearrowleft S^1$ not free

not smooth $\quad$ ($(0,0) \in \mathbb{C}^2$ fixed pt.)

$\boxed{c \neq 0:}$ $\quad H^{-1}(c)/S^1 \cong (\mathbb{C}, \omega_0)$

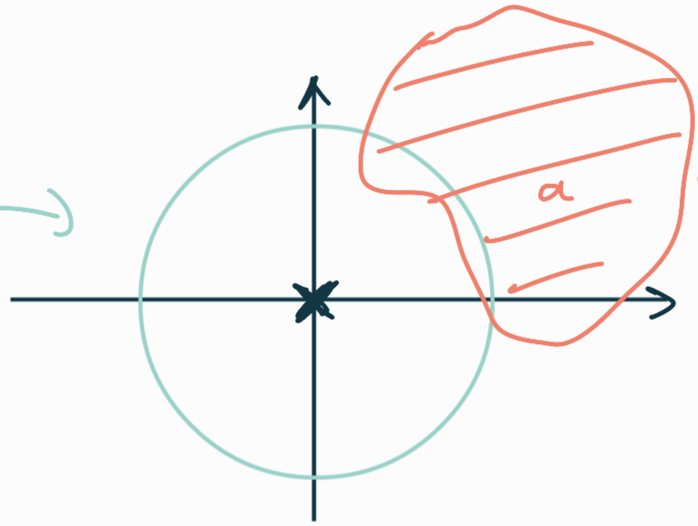
product tori $\subset H^{-1}(c)$ $\longleftrightarrow$ circles $\subset \mathbb{C}$

$T(a_1, a_2) \longleftrightarrow$



$\boxed{c = 0:}$ $\quad \left. H^{-1}(0) \setminus \underbrace{\{(0,0)\}}_{} \right/ S^1 \;\hat{\cong}\; (\mathbb{C}^\times, \omega_0|_{\mathbb{C}^\times})$

$\underbrace{T(a,a) = T_{Cl}(a)}_{}$ $\longleftrightarrow$



$\leftarrow T_{Ch}(a)$

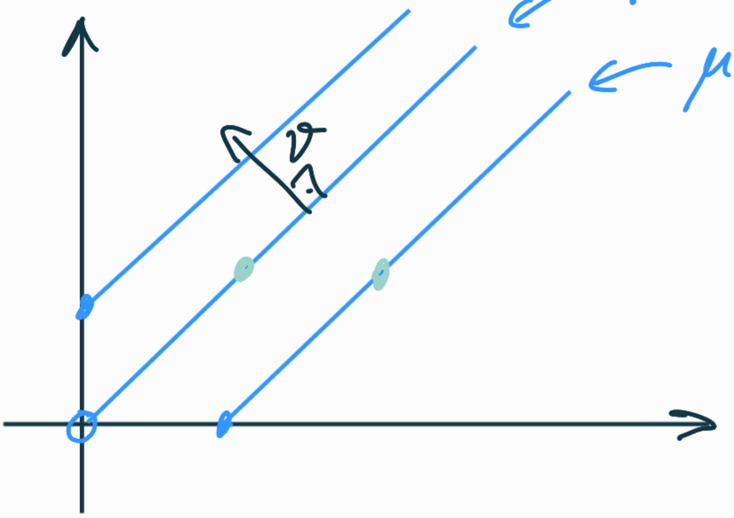
In the toric picture of $\mathbb{C}^2$:

$$\underbrace{\mu : \mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}, \; \mu(z_1, z_2) = (\overset{=\mu_1}{\pi|z_1|^2}, \overset{=\mu_2}{\pi|z_2|^2})}_{}$$

$\underbrace{H = \mu_1 - \mu_2}_{}$

$\mu(H^{-1}(0))$

$\mu(H^{-1}(c))$



fibres of μ = product tori.

<u>Rk</u>: $\quad H = \underline{\langle \mu, v \rangle} \qquad v = (1,-1)$

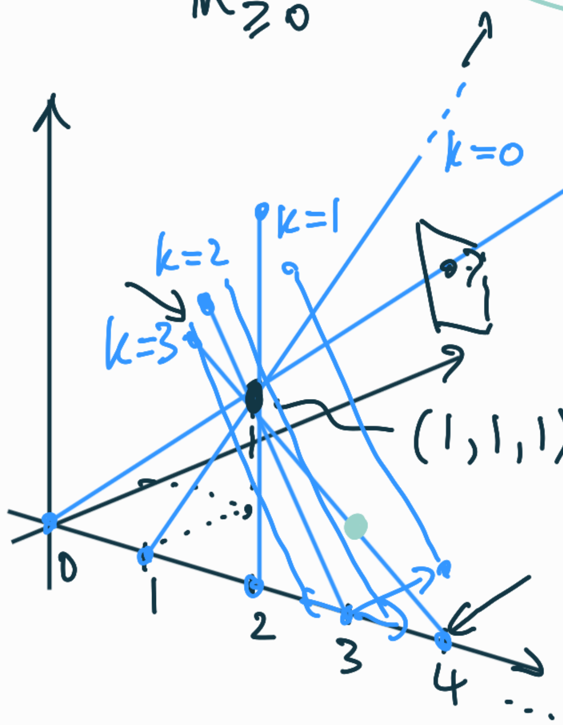
This is a general recipe

$\underbrace{\text{rational segments in } \mathbb{R}^n_{\geq 0} \rightsquigarrow \text{symplectic red. from } \mathbb{C}^n \text{ to surface.}}_{}$

cf. probes by McDuff.

Construction of $\Gamma_k(a_1, a_2) \subset \mathbb{C}^3$:

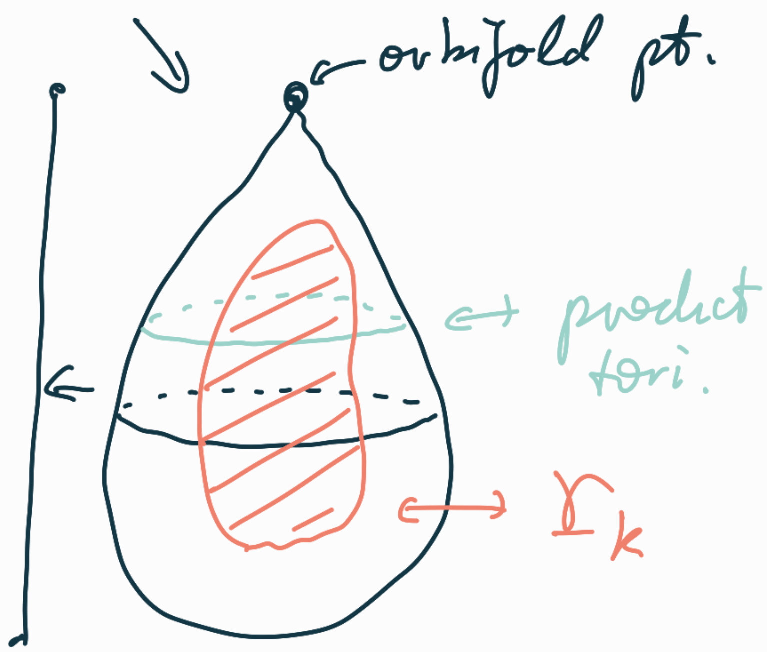
Symplectic reduction on segments w/ direction $\boxed{\begin{pmatrix} -k \\ 1 \\ 1 \end{pmatrix} \in \mathbb{Z}^3}$



$\rightsquigarrow$ Symplectic reduction by T^2-action.

For $\boxed{k \geq 2}$, get orbifold reduced spaces with one puncture:

$\Gamma_k(a_1, a_2)$



<u>Def.</u>: The tori Γ_k are lifts of red curves in the red. spaces.

§ 3. Versal deformations and displacement energy.

(Chekanov '96 see also C.-S. '10, '16; B. '20)

Recall: displacement energy.

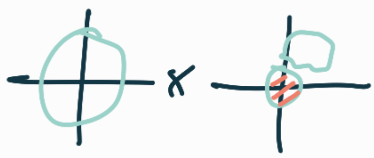
$$e(L) = \inf\{\, \|H\| \mid \phi_1^H(L) \cap L = \emptyset \,\}$$

Hofer norm (= ∞ if empty)

e.g. $e(S^1(a)) = a$

$e(T(a_1 \dots a_n)) = \min\{a_1 \dots a_n\}$

Thm. (Chekanov '98)



(X,ω) tame and J almost complex str. taming ω. $L \subset X$ compact Lagrangian.
Then:

$$e(L) \geq \min\left\{ \begin{array}{l} \text{min. area of } J\text{-curves in } (X,J), \\ \text{min area of } J\text{-disks w/ bdry on } L \end{array} \right\}$$

Unfortunately,

$$e(T_{Cl}(a)) = e(T_{Ch}(a)) = a$$

<u>But</u>: $e(-)$ behaves differently on neighbourhoods of $T_{Cl}(a)$ and $T_{Ch}(a)$ in the space of Lagrangian tori. (Versal def.)

<u>More precisely</u>: $L \subset (X, \omega)$ cpct.

$$\{C^1\text{-small neighb. of } L\} \big/ \sim \;\simeq\; \mathcal{U} \subset H^1(L;\mathbb{R})$$

Ham. isotopies supp. in Weinst. chart. Weinstein's nbhd thm.

Get a function

$$H^1(L;\mathbb{R}) \supset \mathcal{U} \xrightarrow{e(-)} \mathbb{R} \cup \{+\infty\} \quad (*)$$

<u>Def.</u>: The displacement energy germ $\mathcal{E}_L : H^1(L;\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ is the germ of $(*)$ at 0.

Prop: Symplectic invariance:

$$\mathcal{E}_{\phi(L)} \circ \phi|_L^* = \mathcal{E}_L$$

$$\forall \phi \in Symp(X,\omega).$$

$$H^1(T_{Cl}(a);\mathbb{R}) \cong \mathbb{R}^2$$

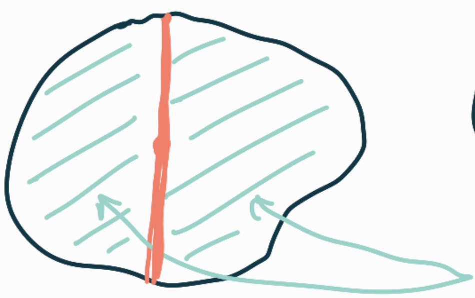
e.g. $\mathcal{E}_{T_{Cl}(a)}(b_1, b_2) = \underbrace{a + \min\{b_1, b_2\}}$

The "versal deformation" of $T_{Cl}(a)$ is just $(b_1, b_2) \mapsto T(a+b_1, a+b_2)$.

$$\underbrace{\mathcal{E}_{T_{Ch}(a)}(c_1, c_2) = a + c_1}$$

Cor.: $T_{Ch}(a)$ is exotic.

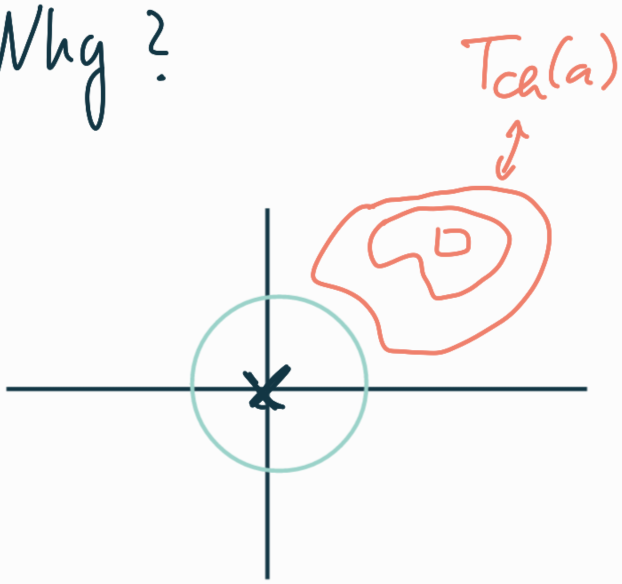
Versal deformation of $T_{Ch}(a)$:



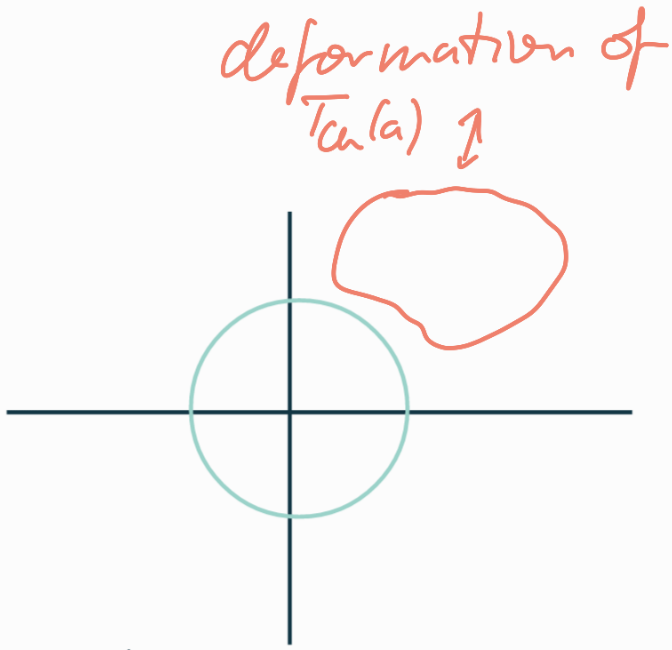
$$\underbrace{\mathcal{U} \subset \overset{\cong \mathbb{R}^2}{H^1(T_{Ch}(a);\mathbb{R})}}$$

(!) It is enough to compute $\mathcal{E}_{T_{Cl}(a)}$ on $\mathcal{U}$: red segment, i.e. to know $e(T(a_1,a_2))$.

Why?

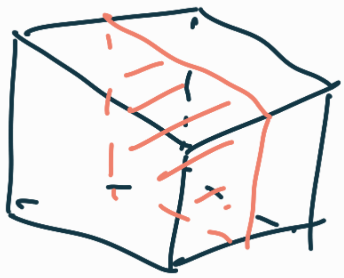


$H^{-1}(0)/S^1$



$H^{-1}(\varepsilon)/S^1$

Same applies to $\Upsilon_k(a_1,a_2)$:



$$\rightsquigarrow \mathcal{E}_{\Upsilon_k(a_1,a_2)}(x,y,z) = \begin{cases} a_2 + \min\{\underbrace{x, x+ky, z}\} & \underline{a_1 = 0 \text{ (monotone)}} \\ a_2 + \min\{x, x+kz\} & \text{non-monotone case} \end{cases}$$

For $k \neq k'$ $\nexists\ \underbrace{\Phi \in GL(3;\mathbb{Z})}$ s.th.

$$\mathcal{E}_{\Upsilon_{k'}} \circ \Phi = \mathcal{E}_{\Upsilon_k}.$$

§4. Embeddings into tame mflds.

Let $\varphi : B^6(R) \hookrightarrow X^6$ Darboux chart.

<u>Goal</u>: Show that for small Υ_k

$$\mathcal{E}_{\varphi(\Upsilon_k)} = \mathcal{E}_{\Upsilon_k}$$

in X^6 in $\mathbb{C}^3$

on an open dense subset.

Compute $e(\varphi(T))$ for T nearby Υ_k.

Pick J tame extending $\underbrace{\varphi_* J_0}$, on $\operatorname{im}\varphi$.

Take $\varepsilon < \min\left\{ \boxed{R/2},\ \underbrace{\{J\text{-curve in } (X,J)\}}_{\text{area of smallest}} \right\}$

and take $\Upsilon_k \subset B^6(\varepsilon)$.

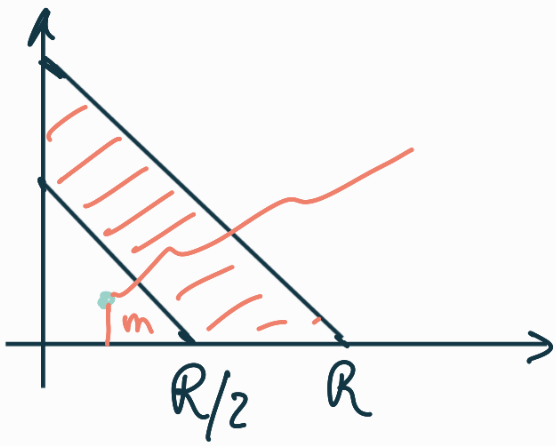
Since $\varepsilon < R/2$, T can be displaced in $B^6(R)$.

$$e(\varphi(T)) \leq e(T).$$

For the other inequality use Chekanov's thm.

1) J-holomorphic disks w/ bdry on $\varphi(T)$

disks contained in im φ are understood.



<u>Claim:</u> minimal J-disk in (X, J)
= minimal J-disk in $B^6(R)$
for all $T \subset B^6(\varepsilon)$.

<u>Prop.</u> (C.-S. '16) If the disk leaves $B^6(R)$, then $\text{area}_\omega > m$.

2) J - hol. curves in (X, J) do not interfere, since $\varepsilon <$ min. area of J-curve in (X, J).

§ 5. Further directions.

1. What about dimension four? Same argument applies to distinguish $\varphi(T_{cl})$, $\varphi(T_{ch})$

   But: Sometimes $\varphi(T_{ch})$ is not exotic in a "meaningful" way:

   Ex.: X^4 toric and $\varphi(T_{ch}) \cong$ toric fibre.

Next best: Construct exotic tori in the nbhd of Lagrangian S^2, $\mathbb{RP}^2$, pinwheels.

(Ongoing discussions with J. Schmitz & J. Hauher)

e.g. L = Lagrangian sphere $\subset X^4$

In a Weinstein chart, have

$T_{cl}, T_{ch}, T_{Pol} \subset T^*S^2$

Show that they are distinct in X^4.

2. In dimension 4, $T_{ch} \subset \mathbb{C}^2$ is a fibre of an almost toric fibration.

<u>Idea</u>: "Extend" Γ_k to fibrations on $\mathbb{C}^3$ should give an idea of how to get higher dimensional ATFs.

(The versal deformation is a local such fibration.)

Thank You!