CH, GW and the geography of tight convex hypersurfaces

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Overview

Convex hypersurfaces are central to contact topology and a hot research topic.

We currently know $\epsilon \sim 0$ in $\dim > 3$: $\exists$ few tools and few well-understood examples.

Today: Characteristic classes don't obstruct tightness in $\dim = 4 + 1$.
Contrasts with the Thurston-Bennequin bound in $\dim = 2 + 1$.
Conjectural picture: $\exists$ lots of tight contact structures in high $\dim$, unlike $\dim = 3$.

Summary for today's lecture:

1. Review what's known and what's not.
2. Investigate geography in $\dim = 4 + 1$. Along the way...
3. New techniques for building interesting convex hypersurfaces.
4. New techniques for computing contact homologies eg. using relative GW.

Covers upcoming sequel to "An algebraic generalization of Giroux's criterion".

Crash course in contact topology

Contact manifold is (M, ξ) of $\dim = 2n + 1$ with $\xi = \ker \alpha$ and $\alpha \wedge d\alpha^n > 0$.

Liouville manifold is (W, β) with $d\beta$ symplectic $\implies$ $(\partial W, \ker \beta)$ contact.
Most important Liouvilles are **Weinstein**, built from handles of ind $\leq \frac{1}{2} \dim W$. Ex:

1. Varieties in $\mathbb{C}^N$ intersect $\mathbb{D}(\rho \gg 0)$ with $\beta = xdy - ydx$.
2. Disk cotangent bundles $(\mathbb{D}^* Q, \beta = pdq)$. Setup for Hamiltonian mechanics on Q.

Filling of (M, ξ) is (W, ω) with $\partial W = M$ and $\omega = d\beta$ Liouville near ∂W.

(M, ξ) are either **tight** or **overtwisted (OT)**:

1. Fillable $\implies$ tight (Gromov, Niederkrüger).
2. $\exists!$ OT ξ for each formal homotopy class of $[\xi]$ on M (Borman-Eliashberg-Murphy).
3. There are many characterizations of OT (Casals-Murphy-Presas).

Important questions:

1. Given a $[\xi]$, are there tight ξ in the class $[\xi]$?
2. Given a (M, ξ), is it tight / fillable?

Convex hypersurfaces powerful tool in $\dim = 3$. Want to understand them in $\dim > 3$!

Convex hypersurfaces I: What are they?

$S \subset M^{2n+1}$ is **convex** if $\exists N(S) = (-\epsilon, \epsilon)_\tau \times S \subset M$, s.t. ξ is τ-invariant. Can assume

$$\alpha|_{N(S)} = f d\tau + \beta, \quad f \in \mathcal{C}^\infty(S), \quad \beta \in \Omega^1(S).$$

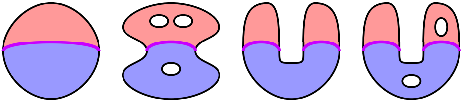
S decomposes into a negative region, a dividing set, and a positive region:

$$S = S^- \cup \Gamma \cup S^+$$

$$S^\pm = \{\pm f > 0\}, \quad \text{on which } \beta^\pm = \pm f^{-1}\beta \text{ is Liouville}$$

$$\Gamma = \{\pm f = 0\}, \quad \text{on which } \alpha \text{ is contact}$$

Our perspective: S is a pair (S^-, S^+) of fillings of (Γ, ξ_Γ) glued along their boundaries.



1. For $n = 1$, determined by simple closed, null-homologous multi-curve Γ.
2. For $n = 2$, much is known about fillings of $(\Gamma^3, \xi_\Gamma) \implies$ know many examples.

Convex hypersurfaces II: Main theorems

Theorem (Giroux, Honda-Huang)

A $\mathcal{C}^0$-generic $S \subset M^{2n+1}$ is convex with the $(S^{\pm}, \beta^{\pm})$ Weinstein.

Theorem (Giroux's criterion for $\dim(S) = 2$)

1. *If $S = \mathbb{S}^2$, then $N(S)$ is tight iff Γ connected.*
2. *If $\chi(S) \leq 0$, then $N(S)$ is tight iff $S^{\pm}$ have no $\mathbb{D}^2$ components.*

$\implies$ ***Thurston-Bennequin bound****, $|e(\xi)[S]| \leq |\chi(S)|$ for tight $N(S)$.*

TB bound $\implies$ Finitely many $H^2(M)$ elements can be $c_1(\xi)$ for a tight ξ on fixed M.

Mantra: In $\dim = 3$, tight ξ are rare! See Colin-Giroux-Honda for details.

Question

Is there a $\dim > 2 + 1$ Giroux criterion or TB bound?

Main results for today

Looking for generalizations of TB, compare invariants of S to Chern numbers of $\xi|_S$. We focus on $\dim S = 4$ and look at χ, σ. Chern numbers are $c_1(\xi)^2, c_2(\xi) = e(\xi)$.

$$\chi = e(\xi) \bmod 2, \quad e(\xi) + \sigma = 0 \bmod 4, \quad c_1(\xi)^2 = 3\sigma + 2e(\xi).$$

Theorem

$\forall (e(\xi), c_1(\xi)^2, \chi, \sigma) \in \mathbb{Z}^4$ as above, $\exists$ connected S of $\dim = 4$ with $(N(S), \xi)$ tight.
$\implies$ "TB difference" $|\chi| - |e(\xi)|$ takes all $2\mathbb{Z}$ values for tight S of $\dim = 4$.
So characteristic classes don't obstruct tightness in $\dim = 4 + 1$.

Outline of the proof:

1. Build some $(N(S), \xi)$ using "divisor pairs" with specific $e(\xi), \sigma$.
2. Use contact homology, $CH \neq 0 \implies$ tight.
   Computations combine relative GW with "Algebraic Giroux Criterion" (AGC).
3. Modify the $(N(S), \xi)$ using handle attachments to get any χ with $e(\xi), \sigma$ fixed.
4. Application of handle attachments preserves $CH \neq 0$.

En route, we'll see some $CH(N(S), \xi) = 0$ ex's via fun GW counts :D

Divisor pairs I: Definition

(X, ω) closed, integral symp. mfld with divisor $D = \mathrm{PD}_X(\omega)$ and neighborhood η.

1. $X \setminus D$ is Liouville. Moreover Weinstein when (X, D) algebraic.
2. $c_1(\eta \to D) = \omega|_D$. $\Gamma = \partial\eta$ a prequantization $\mathbb{S}^1$ bundle, (Γ, ξ_Γ).

A **divisor pair** consists of (X^-, ω^-, D^-), (X^+, ω^+, D^+), and ϕ where

1. $(X^\pm, \omega^\pm)$ are closed symp mflds with divisors $D^\pm$.
2. $\phi : (D^-, \omega^-_{D^-}) \to (D^+, \omega^-_{D^+})$ a symplectomorphism.

From divisor pair, build convex hypersurface $S = S(X^\pm, \omega^\pm, D^\pm, \phi)$ so that

1. $S^\pm = X^\pm \setminus D^\pm$,
2. boundaries are identified by lifting ϕ to the $\mathbb{S}^1$-bundles.

Idea comes from Gompf's fiber sum construction.

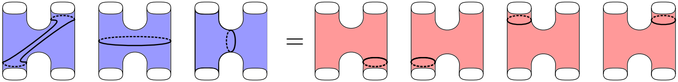
dim $S = 2$: $X^\pm$ closed surfaces, $D^\pm$ points. Every S comes from a divisor pair.

dim $S = 4$: The $\eta(D) \subset X$ are determined entirely by $[D]^2$ self-intersection and $\chi(D)$.

Ex 1: $D_k^- \subset X^- = \mathbb{P}^2, \deg = (2k)$, $D_k^+ \subset X^+ = \mathbb{P}^1 \times \mathbb{P}^1, \deg = (2k, k)$,

$$[D_k^\pm]^2 = 4k^2, \quad \chi(D_k^\pm) = 2k(3 - 2k).$$

$S_k^\pm = X^\pm \setminus D_k^\pm$ are Weinstein with Lefschetz fibrations given by algebraic pencils.
$k = 1$: unique fillings of $L(4,1)$ (McDuff), LFs give lantern relation (Auroux-Smith).



Ex 2: $D_l^- \subset X^- = \Sigma_2 \times \Sigma_2, \deg = (2l, 2l)$, $D_l^+ \subset X^+ = \Sigma_2 \times \Sigma_1, \deg = (4l, l)$,

$$[D_l^\pm]^2 = 8l^2, \quad \chi(D_l^\pm) = -8l(l + 1).$$

For the main theorem, we'll use as building blocks

1. S_k from **Ex 1** series all with $e(\xi) = 1, \sigma = -1$
2. S_l from **Ex 2** series all with $e(\xi) = 4, \sigma = 0$.

CH background

We use contact homology *CH* to verify tightness of our $(N(S), \xi)$.

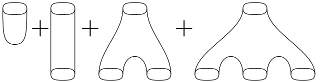
Theorem (Bourgeois-van Koert, Eliashberg, M.L. Yau)

$CH(M, \xi) \neq 0 \implies (M, \xi)$ tight.

On chain level, get graded-comm DGA (CC, ∂) from V the $\mathbb{Q}$ space of Reeb orbits

$$CC = \mathcal{S}(V) = \bigoplus_{0}^{\infty} V^{\otimes k} / \sim, \quad xy \sim (-1)^{|x|\cdot|y|} yx, \quad \partial(xy) = (\partial x)y + (-1)^{|x|} x(\partial y).$$

∂ counts holo-curves in $\mathbb{R}_s \times M$ and breaks up as $\partial = \sum_0^\infty \partial_k$

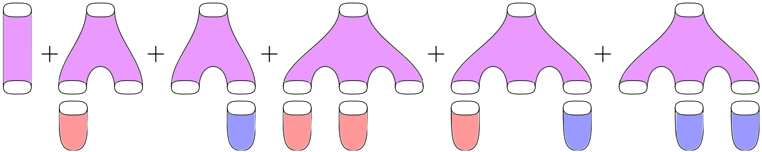
$\partial\gamma =$  $+ \dots$

Liouville fillings give **augmentations**, $\epsilon : (CC, \partial) \to (\mathbb{Q}, \partial_{\mathbb{Q}} = 0)$.

Bilinearization

For $S = S^- \cup \Gamma \cup S^+$ get two augmentations $\epsilon^-, \epsilon^+ : CC_\Gamma \to \mathbb{Q}$.

Use $\vec{\epsilon} = (\epsilon^-, \epsilon^+)$ and $\partial_{\Gamma, k \geq 1}$ to define $\deg = -1$ maps on $\widehat{V} = V[1]$

$\partial_1^{\vec{\epsilon}} \widehat{\gamma} =$  $+ \dots$

$$\partial_1^{\vec{\epsilon}} : \widehat{V} \to \widehat{V}, \quad \partial_0^{\vec{\epsilon}} : \widehat{V} \to \mathbb{Q}, \quad \partial_0^{\vec{\epsilon}} \widehat{v} = \epsilon^+ v - \epsilon^- v$$

Get commutative versions of LCH objects (Bourgeois-Chantraine + Bourgeois-Galant):

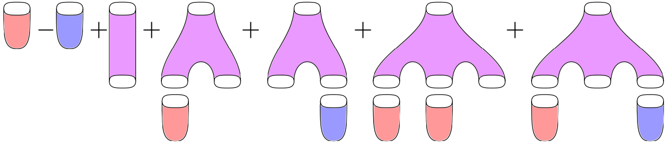
1. $(\partial_1^{\vec{\epsilon}})^2 = 0$ and defines **bilinearized homology** $H^{\vec{\epsilon}} = H(\widehat{V}, \partial_1^{\vec{\epsilon}})$
2. $\partial_0^{\vec{\epsilon}} \partial_1^{\vec{\epsilon}} = 0 \implies \partial_0^{\vec{\epsilon}}$ induces map $H^{\vec{\epsilon}} \to \mathbb{Q}$, the **fundamental class**.

$$\implies (\widehat{CC} = \mathcal{S}(\widehat{V}), \partial^{\vec{\epsilon}} = \partial_0^{\vec{\epsilon}} + \partial_1^{\vec{\epsilon}})$$

is a free CDGA, called the **bilinearized DGA**.

The Algebraic Giroux Criterion

For $S = S^{-} \cup \Gamma \cup S^{+}$ get augmentations $\epsilon^{-}, \epsilon^{+} : CC_{\Gamma} \to \mathbb{Q}$ and bilinearized algebra...

$\partial^{\vec{\epsilon}}\widehat{\gamma} =$  $+ \dots$

Theorem (Algebraic Giroux Criterion (AGC))

$\exists$ *geometric data so that* $(CC_{N(S)}, \partial_{N(S)})$ *is the bilinearized algebra. Therefore...* $CH(N(S), \xi) \neq 0$ *iff fundamental class* $\partial_0^{\vec{\epsilon}} = 0$ *on* $H(\widehat{V}, \partial_1^{\vec{\epsilon}})$.

Sounds complicated... The $\epsilon^{\pm}$ are usually difficult to compute :(

For convex hypersurfaces determined by divisor pairs...

1. have easy formulas for indices, $\implies$ can often say $\epsilon^{\pm} = 0$ for index reasons.
2. otherwise can use relative GW, often boiling down to classical counting :D

Plane sphere correspondence I

To compute CH we need to count augmentation planes in $X \setminus D$.

$D \subset X^{2n}$ divisor with neighborhood η, $\partial\eta = \Gamma$. $f \in \mathcal{C}^\infty(D)$ Morse. Get Reeb with

$$\text{closed orbits} = \{\gamma_p^{\mathrm{cm}} : \ p \in \mathrm{Crit}(f)\}, \quad \text{all cover } \mathbb{S}^1 \text{ fibers of } \Gamma \to D.$$

Basis of T_pD gives framing of orbits $\gamma_p^{\mathrm{cm}}, p \in \mathrm{Crit}(f)$ so that

$$\mathrm{CZ}(\gamma_p^{\mathrm{cm}}) = \mathrm{ind}_{Mo}(f,p) - n + 1, \quad |\gamma_p^{\mathrm{cm}}| = \mathrm{ind}_{Mo}(f,p) - 2.$$

For a $u : \mathbb{C} \to X \setminus D$ asymptotic to γ_p^{cm}, get $\overline{u} : \mathbb{P}^1 \to X$ by filling in the point at ∞.

Use GW index formula to relate $\mathrm{ind}(u), \mathrm{ind}(\overline{u})$, and cm by

$$\boxed{\mathrm{cm} = [D] \cdot [\overline{u}], \quad \mathrm{ind}(u) = 2(c_1(X) - [D])[\overline{u}] + \mathrm{ind}_{Mo}(f,p) - 2.}$$

This is super easy to use when D is algebraic inside of Kähler X.

Plane sphere correspondence II

$D \subset X^{2n}$ divisor neighborhood η and $\partial\eta = \Gamma$. $f \in \mathcal{C}^{\infty}(D)$ Morse.

$$u : \mathbb{C} \to X \setminus D, \quad \infty \to \gamma_p^{\mathrm{cm}},\ p \in \mathrm{Crit}(f) \quad \rightsquigarrow \quad \overline{u} : \mathbb{P}^1 \to X$$

$$\boxed{\mathrm{cm} = [D] \cdot [\overline{u}], \quad \mathrm{ind}(u) = 2(c_1(X) - [D])[\overline{u}] + \mathrm{ind}_{Mo}(f, p) - 2.}$$

Often formula tells us there are no $\mathrm{ind}(u) = 0$ augmentation planes in $X \setminus D$.

Suppose f has unique p_{min}, p_{max} of $\mathrm{ind}_{Mo}(f, p_{min}) = 0$ and $\mathrm{ind}_{Mo}(f, p_{max}) = 2n - 2$.

Theorem (Simplified plane-sphere correspondence)

Suppose $\mathcal{M}$ involved are transversely cut out. Then

1. *$\#(u)$ asymptotic to $\gamma_{p_{min}}^{\mathrm{cm}}$ is GW count of $\overline{u}$ touching D only at $p_{\min} \in D$.*
2. *$\#(u)$ asymptotic to $\gamma_{p_{max}}^{\mathrm{cm}}$ is GW count of $\overline{u}$ touching D exactly once.*
3. *Covering multiplicity $\mathrm{cm} \iff$ order of contact for $\overline{u} \cap D$.*

Comes from Diogo-Lisi's *SH* computations for divisor complements.

Simplified case above enough to compute in $n = 2$ case of interest today.

Computational examples

Ex 1: S_k from $D_k^- \subset X^- = \mathbb{P}^2, \deg = (2k)$, $D_k^+ \subset X^+ = \mathbb{P}^1 \times \mathbb{P}^1, \deg = (2k, k)$,

$$[D_k^\pm]^2 = 4k^2, \quad \chi(D_k^\pm) = 2k(3-2k).$$

Ex 2: S_l from $D_l^- \subset X^- = \Sigma_2 \times \Sigma_2, \deg = (2l, 2l)$, $D_l^+ \subset X^+ = \Sigma_2 \times \Sigma_1, \deg = (4l, l)$,

$$[D_l^\pm]^2 = 8l^2, \quad \chi(D_l^\pm) = -8l(l+1).$$

Theorem

$CH(N(S_k), \xi_k) \neq 0 \iff k \geq 3$. $CH(N(S_l), \xi_l) \neq 0$ for all l.

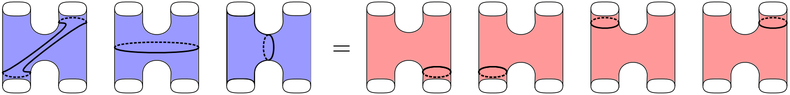
The S_l case is easy. $\pi_2(\Sigma_{g\geq 1}) = 0 \implies \epsilon^\pm = 0$.

We'll only need that some of the $(N(S_k), \xi_k)$ are tight to prove our geography theorem.

But the $k = 1, 2$ cases are too fun to skip :D

Scenic detour: Computation $k = 1$

$D_1^- \subset X^- = \mathbb{P}^2, \deg = (2)$, $D_1^+ \subset X^+ = \mathbb{P}^1 \times \mathbb{P}^1, \deg = (2, 1)$. Here $D_1^\pm \simeq \mathbb{P}^1$.



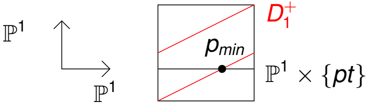
Orbit γ^1_{min} has least action. So $\partial^{\vec{\epsilon}}\gamma^1_{min}$ counts only augmentation planes.

There can be no u^- because for any $\overline{u}$, $\mathrm{cm} = [D^-] \cdot [\overline{u}] \geq 2$.

For u^+ with $[\overline{u}^+] = a[\mathbb{P}^1 \times \{pt\}] + b[\{pt\} \times \mathbb{P}^1]$, want $\mathrm{cm} = 1, \mathrm{ind}(u^+) = 0$ from

$$\mathrm{cm} = a + 2b, \quad \mathrm{ind}(u) = 2a - 2 \implies [\overline{u}^+] = [\mathbb{P}^1 \times \{pt\}].$$

There is **exactly one** such $[\overline{u}^+]$ passing through $p_{min} \in D_1^+$.



So $\partial^{\vec{\epsilon}}\widehat{\gamma}^1_{p_{min}} = 1 \implies CH(N(S_1), \xi) = 0$.

Scenic detour: Computation $k = 2$

$D_2^- \subset X^- = \mathbb{P}^2, \deg = (4)$, $D_2^+ \subset X^+ = \mathbb{P}^1 \times \mathbb{P}^1, \deg = (4, 2)$. Here $D_2^\pm \simeq \Sigma_{g=3}$.

$\partial_{CH(\Gamma)} = 0$. $CH(N(S))$ differential counts only aug planes. Study γ^2_{max}.

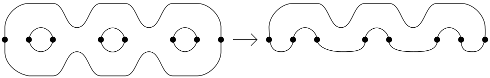
There can be no u^- because for any $\overline{u}$, $\text{cm} = [D^-] \cdot [\overline{u}] \geq 4$.

For u^+ with $[\overline{u}^+] = a[\mathbb{P}^1 \times \{pt\}] + b[\{pt\} \times \mathbb{P}^1]$, want $\text{cm} = 2, \text{ind}(u^+) = 0$ from

$$\text{cm} = 2a + 4b, \quad \text{ind}(u) = -4b \implies [\overline{u}^+] = [\mathbb{P}^1 \times \{pt\}].$$

Generic line $(1, 0)$ touches $D_2^+ \pitchfork$ in two points. Special lines touch D_2^+ **once with multiplicity two**! Special lines $\mathbb{P}^1 \times \{z_2\} \iff$ critical points $(z_1, z_2) \in D_2^+$ of

$$D_2^+ \simeq \Sigma_{g=3} \xrightarrow{\phi} \mathbb{P}^1, \quad (z_1, z_2) \mapsto z_2, \quad \deg(\phi) = 2.$$



We can count with Riemann-Hurwitz:

$$\partial^{\vec{\epsilon}} \widehat{\gamma}^2_{p_{max}} = \#(\text{special lines}) = \#(\text{Crit}(\Sigma_{g=3} \to \mathbb{P}^1)) = 8 \implies CH(N(S_2), \xi) = 0.$$

Summary and setup for geography theorem

In remaining cases, $\mathrm{ind}(u)$ computations show there is nothing to count. So...

Ex 1: S_k from $D_k^- \subset X^- = \mathbb{P}^2, \deg = (2k)$, $D_k^+ \subset X^+ = \mathbb{P}^1 \times \mathbb{P}^1, \deg = (2k, k)$

$$\sigma(S_k) = -1, \quad e(\xi)[S_k] = 1, \quad CH(N(S_k), \xi_k) \neq 0 \text{ for } k \geq 3$$

Ex 2: S_l from $D_l^- \subset X^- = \Sigma_2 \times \Sigma_2, \deg = (2l, 2l)$, $D_l^+ \subset X^+ = \Sigma_2 \times \Sigma_1, \deg = (4l, l)$,

$$\sigma(S_l) = 0, \quad e(\xi)[S_l] = -4, \quad CH(N(S_l), \xi_l) \neq 0.$$

In general characteristic numbers satisfy

$$\chi = e(\xi) \bmod 2, \quad e(\xi) + \sigma = 0 \bmod 4, \quad c_1(\xi)^2 = 3\sigma + 2e(\xi).$$

and we want to prove...

Theorem

$\forall (e(\xi), c_1(\xi)^2, \chi, \sigma) \in \mathbb{Z}^4$ as above, $\exists$ connected S of $\dim = 4$ with $(N(S), \xi)$ tight.

Getting the correct σ, $e(\xi)$

$$\chi = e(\xi) \bmod 2, \quad e(\xi) + \sigma = 0 \bmod 4, \quad c_1(\xi)^2 = 3\sigma + 2e(\xi).$$

Theorem

$\forall (e(\xi), c_1(\xi)^2, \chi, \sigma) \in \mathbb{Z}^4$ *as above,* $\exists$ *connected* S *of* $\dim = 4$ *with* $(N(S), \xi)$ *tight.*

Ex 1: $\boxed{\sigma(S_k) = -1}$, $e(\xi)[S_k] = 1$, $CH(N(S_k), \xi_k) \neq 0$ for $k \geq 3$.

Ex 2: $\sigma(S_l) = 0$, $\boxed{e(\xi)[S_l] = -4}$, $CH(N(S_l), \xi_l) \neq 0$.

To get $\sigma = \sigma_0$ and $e(\xi) = e_0$ as desired take $\sqcup$ of

1. $|\sigma_0|$ copies of $S_{k \geq 3}$ with $\mp \operatorname{sgn}(\sigma_0)$ orientation.
2. $|e_0 + e(\xi)[S_k]|/4$ copies of S_l with $\mp \operatorname{sgn}(\cdots)$ orientation.

Result is tight because $CH(\sqcup N(S_i)) = \bigotimes CH(N(S_i)) \neq 0$.

We want the result to be connected with specific χ. $c_1(\xi)^2$ is determined. We will

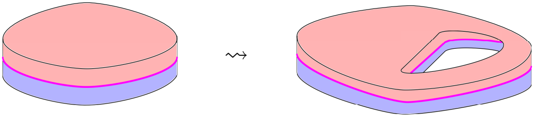
1. add handles to this disjoint union to make connected and correct χ, and
2. show that handle addition preserves $CH \neq 0$.

Contact handle attachments

In $(N(S) = [-1,1]_\tau \times S, \xi)$ neighborhood of $\{1\} \times \Gamma$ looks like

$$[-1,1]_t \times [0,1]_s \times \Gamma, \quad \alpha = dt + e^s \alpha_\Gamma$$

with corners rounded. So it's the contactization of symplectization of $(\Gamma^{2n-1}, \alpha_\Gamma)$.
Contactization of $\mathrm{ind} = i \leq n, \dim = 2n$ Weinstein handle H_w is a **contact handle** H_c.
Can attach H_c to the boundary of any (M, ξ), when ∂M is convex.



After rounding corners, convex boundary is changed by

$$S = (S^-, S^+) \rightsquigarrow S_H = (S^- \cup H_w, S^+ \cup H_w), \quad \chi \rightsquigarrow \chi + 2\,\mathrm{ind}(H_w).$$

σ and $e(\xi)$ unaffected as they are cobordism/homological invariants.

Theorem (implicit in Colin-Ghiggini-Honda-Hutchings)

Adding an $\mathrm{ind} \leq n$ contact handle to the boundary of a (M, ξ) leaves CH unaffected.

Wrapping up

$$\chi = e(\xi) \bmod 2, \quad e(\xi) + \sigma = 0 \bmod 4, \quad c_1(\xi)^2 = 3\sigma + 2e(\xi).$$

Theorem

$\forall (e(\xi), c_1(\xi)^2, \chi, \sigma) \in \mathbb{Z}^4$ as above, $\exists$ connected S of $\dim = 4$ with $(N(S), \xi)$ tight.

We already have $CH \neq 0$ convex hypersurface S with desired $\sigma, e(\xi)$.
To make connected connected add 1-handles. To correct χ add $\text{ind} = 1, 2$ handles.

Need to see that S_H after handle addition has $CH \neq 0$.

1. Inclusion $N(S) \to N(S) \cup H_c$ induces iso on CH. $CH(N(S) \cup H_c) \neq 0$.
2. $N(S_h) \subset N(S) \cup H_c$ induces $CH(N(S_H)) \to CH(N(S) \cup H_c)$ by CGHH.
3. Unital algebra morphism $\implies CH(N(S_H)) \neq 0$.

That completes the proof!

Epilogue

$$\chi = e(\xi) \bmod 2, \quad e(\xi) + \sigma = 0 \bmod 4, \quad c_1(\xi)^2 = 3\sigma + 2e(\xi).$$

Theorem

$\forall (e(\xi), c_1(\xi)^2, \chi, \sigma) \in \mathbb{Z}^4$ *as above,* $\exists$ *connected* S *of* $\dim = 4$ *with* $(N(S), \xi)$ *tight.*

Broader context:

1. See also results of Mori on non-convex surfaces violating a *TB* bound in high dim.
2. Fits into POV of Bowden-Gironella-Moreno-Zhou: $\exists$ many tight ξ in high dim!
3. Conjecture: $\nexists$ generalized *TB* bound involving characteristic numbers in $\dim \geq 5$.
4. For today's strategy to general, need more divisor pairs in high dim...

Here's one interesting high dim example studied from the symplectic POV by Oba:

1. $D^- \subset X^- = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \deg = (1,1,1)$,
2. $X^+ = \deg(1,1) \subset \mathbb{P}^2 \times \mathbb{P}^2$, $D = \deg(1,1) \subset X^+$.
3. $D^\pm \simeq \mathbb{P}^2 \# 3\overline{\mathbb{P}}^2$ are $\deg = 6$ del Pezzo surfaces.
4. I can show $CH(N(S), \xi) = 0$ by GW count.

Let me know if you have further $\dim_{\mathbb{C}} > 2$ examples!

Thanks for having me!

See me in next Symplectix (Zoom-able) to hear about contact submanifolds in high dim!

References:

1. Auroux and Smith, "*Lefschetz pencils, branched covers, and symplectic invariants*"
2. Avdek, "*An algebraic generalization of Giroux's criterion*"
3. Bennequin, "*Entrelacements et equations de Pfaff*"
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