A metric on the contactomorphism group of an orderable contact manifold

Lukas Nakamura
October 30, 2023
Contact manifolds

Let $(M^{2n+1}, \alpha)$ be a connected strict contact manifold, i.e. $\alpha$ is a 1-form satisfying $\alpha \wedge (d\alpha)^n \neq 0$.

$\text{Cont}_0(M, \xi) \subseteq \text{Diff}(M)$ denotes the identity component of the group of compactly supported contactomorphisms $\phi$ on $(M, \xi)$, i.e. $\phi_*\xi = \xi$, where $\xi := \ker \alpha$.

Given a contact isotopy $\phi_t$, we can associate to it its Hamiltonian $H_t = \alpha(\partial_t \phi_t) : M \to \mathbb{R}$.

Conversely, given a time-dependent function $H_t : M \to \mathbb{R}$, there exists a unique contact isotopy $\phi^H_t$ starting at the identity with Hamiltonian $H_t$.

The contact isotopy $\phi^\alpha_t$ corresponding to $H_t \equiv 1$ is called the Reeb flow.
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The contact isotopy $\phi_t^\alpha$ corresponding to $H_t \equiv 1$ is called the Reeb flow.
If $H_t \geq 0$ (resp. $H_t > 0$), then $\phi_t^H$ is called non-negative (resp. positive).

We write $\phi_0 \leq \phi_1$ (resp. $\phi_0 \ll \phi_1$) if there exists a non-negative (resp. positive) contact isotopy from $\phi_0$ to $\phi_1$.

$(M, \xi)$ is strongly orderable if $\leq$ defines a partial order on $\text{Cont}_0(M, \xi)$.

**Lemma (Eliashberg-Polterovich, 2000)**

If $M$ is compact, $(M, \xi)$ is strongly orderable if and only if there does not exist a positive loop of contactomorphisms.
Positive paths and orderability

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**Lemma (Eliashberg-Polterovich, 2000)**

If $M$ is compact, $(M, \xi)$ is strongly orderable if and only if there does not exists a positive loop of contactomorphisms.
If $M$ is compact, we define

$$I_{\phi_0, \phi_1}^\ll := \{ \phi \in \text{Cont}_0(M, \xi) | \phi_0 \ll \phi \ll \phi_1 \}.$$  

(1)

- These intervals generate a topology $O_{\ll}$, called interval topology.
- If $(M, \xi)$ is not strongly orderable, $O_{\ll} = \{ \emptyset, \text{Cont}_0(M, \xi) \}$.

**QUESTION (CHERNOV-NEMIROVSKI, 2020)**

If $(M, \xi)$ is strongly orderable, is $O_{\ll}$ Hausdorff?
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Energies of contact isotopies

For $H_t : M \to \mathbb{R}$, define

$$\| \phi_t^H \|^\alpha_{SH} := \| H_t \| := \int_0^1 \max_M |H_t| \, dt.$$  \hfill (2)

For $\phi_0, \phi_1 \in \text{Cont}_0(M, \xi)$, define their Shelukhin-Hofer distance as

$$d_{SH}^\alpha(\phi_0, \phi_1) := \inf_{H_t} \{ \| \phi_t^H \|^\alpha | \phi_1^H \phi_0 = \phi_1 \}. \hfill (3)$$

**Theorem (Shelukhin, 2017)**

$d_{SH}^\alpha$ is a non-degenerate, right-invariant, and left-natural metric (i.e.

$$d_{SH}^\alpha(\psi \phi_0, \psi \phi_1) = d_{SH}^{\psi^* \alpha}(\phi_0, \phi_1).$$
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Energies of contact isotopies

For $H_t : M \to \mathbb{R}$, define

$$\|H_t\|_{\pm} := \int_0^1 \max_M \{\pm H_t\} dt.$$  \hspace{1cm} (4)

For $\phi_0, \phi_1 \in \text{Cont}_0(M, \xi)$, we define

$$\delta_{\pm}^\alpha(\phi_0, \phi_1) := \inf_{H_t} \{\|H_t\|_{\pm}| \phi_0^H \phi_0 = \phi_1\}.$$  \hspace{1cm} (5)

In fact,

$$\delta^-\alpha(\phi_0, \phi_1) = \inf\{\varepsilon \in \mathbb{R}| \exists H_t : M \to [-\varepsilon, \infty) : \phi_0^H \phi_0 = \phi_1\},$$

$$\delta^+\alpha(\phi_0, \phi_1) = \delta^-\alpha(\phi_1, \phi_0) = \inf\{\varepsilon \in \mathbb{R}| \exists H_t : M \to (-\infty, \varepsilon] : \phi_0^H \phi_0 = \phi_1\}.$$  \hspace{1cm} (6)
Energies of contact isotopies

For $H_t : M \to \mathbb{R}$, define

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$$\delta_{\pm}^{+}(\phi_0, \phi_1) = \delta_{\pm}^{-}(\phi_1, \phi_0) = \inf\{\varepsilon \in \mathbb{R} \mid \exists H_t : M \to (-\infty, \varepsilon] : \phi_1^H \phi_0 = \phi_1\}.$$  \hfill (6)
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$$\delta_\pm^\alpha(\phi_0, \phi_1) := \inf_{H_t} \{\|H_t\|_\pm | \phi^H_1 \phi_0 = \phi_1\}.$$ \hspace{1cm} (5)

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$$\delta^+_\alpha(\phi_0, \phi_1) = \delta^-_\alpha(\phi_1, \phi_0) = \inf \{\varepsilon \in \mathbb{R} | \exists H_t : M \to (-\infty, \varepsilon] : \phi^H_1 \phi_0 = \phi_1\}.$$ \hspace{1cm} (6)
Energies of contact isotopies

We define

\[ d_\alpha(\phi_0, \phi_1) := \max\{\delta_\alpha^+(\phi_0, \phi_1), \delta_\alpha^-(\phi_0, \phi_1), 0\}. \]  

(7)

**Remark 0.1**

- In the case that \( M \) is compact, \( \delta_\alpha^\pm \) and \( d_\alpha \) appeared first in the PhD thesis of Arlove.
- The so-called *Lorentzian distance function* \( \tau_\alpha = \max\{-\delta_\alpha^-, 0\} \) was introduced and studied by (Hedicke, 2021).
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Main result

**Theorem (N., 2023)**

d_α is a right-invariant and left-natural pseudo-metric, which is either non-degenerate or vanishes identically, such that

- if $M$ is compact, $(M, \xi)$ is strongly orderable if and only if $d_\alpha$ is non-degenerate,
- $\mathcal{O}_\ll = \mathcal{O}_{d_\alpha}$ for compact $M$,
- $d_\alpha \leq d_{SH}$. 
Legendrian submanifolds

$L^n \subseteq M^{2n+1}$ is Legendrian if $TL \subseteq \xi$.

- Isotopies of Legendrian submanifolds are induced by ambient contact isotopies.  
  \( \Rightarrow \delta_{\alpha}^\pm \) induce distances \( \delta_{\alpha}^{\pm, L} \) on the Legendrian isotopy class of \( L \).

- We define \( d_{\alpha}^L := \max\{\delta_{\alpha}^{+ L}, \delta_{\alpha}^{- L}, 0\} \).

- For compact \( L \), \( L \) is strongly orderable if there does not exists a positive loop of Legendrians starting at \( L \).
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  \[ \Rightarrow \delta^\pm_\alpha \text{ induce distances } \delta^{\pm,L}_\alpha \text{ on the Legendrian isotopy class of } L. \]

- We define $d^L_\alpha := \max\{\delta^+_{\alpha,L}, \delta^-_{\alpha,L}, 0\}$.

- For compact $L$, $L$ is strongly orderable if there does not exists a positive loop of Legendrians starting at $L$. 
$L^n \subseteq M^{2n+1}$ is **Legendrian** if $TL \subseteq \xi$.

- Isotopies of Legendrian submanifolds are induced by ambient contact isotopies. 
  \[ \Rightarrow \delta^\pm_{\alpha} \text{ induce distances } \delta^\pm_{\alpha} L \text{ on the Legendrian isotopy class of } L. \]
- We define $d^L_\alpha := \max\{\delta^+_\alpha L, \delta^-_\alpha L, 0\}$.
- For compact $L$, $L$ is **strongly orderable** if there does not exist a positive loop of Legendrians starting at $L$. 
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- For compact $L$, $L$ is **strongly orderable** if there does not exist a positive loop of Legendrians starting at $L$. 
Main result for Legendrians

**THEOREM (N., 2023)**

\(d^L_\alpha\) is a right-invariant and left-natural pseudo-metric, such that

- if \(L\) is connected, \(d^L_\alpha\) is either non-degenerate or vanishes identically,
- if \(L\) is compact and connected, \(L\) is strongly orderable if and only if \(d^L_\alpha\) is non-degenerate,
- \(O^L_\ll = O_{d^L_\alpha}\) for compact \(L\),
- \(d^L_\alpha \leq d_{SCH}\).
Universal covers

Remark

There are analogous results on the universal covers of $\text{Cont}_0(M, \xi)$ and Legendrian isotopy classes, if we replace positive loop by contractible positive loop and non-degenerate by the distance of two elements is positive if the underlying contactomorphisms/Legendrians are different.
Relation to the Reeb flow

**PROPOSITION**

If \((M, \xi)\) is strongly orderable,

\[
\delta_{\alpha}^{\pm}(\phi_0, \phi_t^\alpha \phi_1) = \delta_{\alpha}^{\pm}(\phi_0, \phi_1) \pm t.
\] (8)

If \(L\) is strongly orderable,

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\delta_{\alpha}^{\pm, L}(L_0, \phi_t^\alpha (L_1)) = \delta_{\alpha}^{\pm, L}(L_0, L_1) \pm t.
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**REMARK**

For \(\tau_\alpha\), this was first shown by (Hedicke, 2021).
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Relation to the Reeb flow

Remark

- \((M, \xi)\) is strongly orderable if and only if \(d_{SH}(Id_M, \phi_t^\alpha) = |t|\).
- \(L\) is strongly orderable if and only if \(d_{SCH}(L, \phi_t^\alpha(L)) = |t|\).

Theorem (N., 2021)

If \(L\) admits a loose chart of size \(c > 0\) and the Reeb flow is complete, then \(d_{SCH}(L, \phi_t^\alpha(L)) \leq c\).

Corollary

If \(L\) admits a loose chart of size \(c > 0\) and the Reeb flow is complete, \(L\) admits a positive loop of energy less than \(2c + \varepsilon\) (\(\varepsilon > 0\)).
Relation to the Reeb flow

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Spectrality

Remark

In the case that $M$ is closed, or that $L$ is closed and the Reeb flow is complete, the two main results were independently proven by (Allais-Arlove, 2023).

They also showed that $\pm \delta^\pm_{\alpha} (L_0, L_1)$ takes values in the spectrum of actions of Reeb chords from $L_0$ to $L_1$. 
Thank you!