A metric on the contactomorphism group of an orderable contact manifold

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Let (M^{2n+1}, α) be a connected strict contact manifold, i.e. α is a 1-form satisfying $\alpha \wedge (d\alpha)^n \neq 0$.

$\mathrm{Cont}_0(M, \xi) \subseteq \mathrm{Diff}(M)$ denotes the identity component of the group of compactly supported contactomorphisms ϕ on (M, ξ), i.e. $\phi_*\xi = \xi$, where $\xi := \ker \alpha$.

Given a contact isotopy ϕ_t, we can associate to it its *Hamiltonian* $H_t = \alpha(\partial_t \phi_t) : M \to \mathbb{R}$.

Conversely, given a time-dependent function $H_t : M \to \mathbb{R}$, there exists a unique contact isotopy ϕ_t^H starting at the identity with Hamiltonian H_t.

The contact isotopy ϕ_t^α corresponding to $H_t \equiv 1$ is called the *Reeb flow*.

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Positive paths and orderability

If $H_t \geq 0$ (resp. $H_t > 0$), then ϕ_t^H is called *non-negative* (resp. *positive*).

We write $\phi_0 \leq \phi_1$ (resp. $\phi_0 \ll \phi_1$) if there exists a non-negative (resp. positive) contact isotopy from ϕ_0 to ϕ_1.

(M, ξ) is *strongly orderable* if $\leq$ defines a partial order on $\mathrm{Cont}_0(M, \xi)$.

Lemma (Eliashberg-Polterovich, 2000)

If M is compact, (M, ξ) is strongly orderable if and only if there does not exists a positive loop of contactomorphisms.

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Interval topology

If M is compact, we define

$$I^{\ll}_{\phi_0,\phi_1} := \{\phi \in \mathrm{Cont}_0(M,\xi) \,|\, \phi_0 \ll \phi \ll \phi_1\}. \tag{1}$$

- These intervals generate a topology $\mathcal{O}_{\ll}$, called *interval topology*.
- If (M,ξ) is not strongly orderable, $\mathcal{O}_{\ll} = \{\emptyset, \mathrm{Cont}_0(M,\xi)\}$.

QUESTION (CHERNOV-NEMIROVSKI, 2020)

If (M,ξ) is strongly orderable, is $\mathcal{O}_{\ll}$ Hausdorff?

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Energies of contact isotopies

For $H_t : M \to \mathbb{R}$, define

$$\|\phi_t^H\|_{SH}^{\alpha} := \|H_t\| := \int_0^1 \max_M |H_t| dt. \tag{2}$$

For $\phi_0, \phi_1 \in \mathrm{Cont}_0(M, \xi)$, define their *Shelukhin-Hofer distance* as

$$d_{SH}^{\alpha}(\phi_0, \phi_1) := \inf_{H_t} \{ \|\phi_t^H\|_{\alpha} | \, \phi_1^H \phi_0 = \phi_1 \}. \tag{3}$$

THEOREM (SHELUKHIN,2017)

d_{SH}^{α} is a non-degenerate, right-invariant, and left-natural metric (i.e. $d_{SH}^{\alpha}(\psi\phi_0, \psi\phi_1) = d_{SH}^{\psi^*\alpha}(\phi_0, \phi_1)$).

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For $H_t : M \to \mathbb{R}$, define

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In fact,

$$\begin{aligned} \delta_\alpha^-(\phi_0, \phi_1) &= \inf\{\varepsilon \in \mathbb{R} \mid \exists H_t : M \to [-\varepsilon, \infty) : \phi_1^H \phi_0 = \phi_1\}, \\ \delta_\alpha^+(\phi_0, \phi_1) = \delta_\alpha^-(\phi_1, \phi_0) &= \inf\{\varepsilon \in \mathbb{R} \mid \exists H_t : M \to (-\infty, \varepsilon] : \phi_1^H \phi_0 = \phi_1\}. \end{aligned} \tag{6}$$

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We define

$$d_\alpha(\phi_0, \phi_1) := \max\{\delta_\alpha^+(\phi_0, \phi_1), \delta_\alpha^-(\phi_0, \phi_1), 0\}. \tag{7}$$

REMARK 0.1

- In the case that M is compact, $\delta_\alpha^\pm$ and d_α appeared first in the PhD thesis of Arlove.
- The so-called *Lorentzian distance function* $\tau_\alpha = \max\{-\delta_\alpha^-, 0\}$ was introduced and studied by (Hedicke, 2021).

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Main result

Theorem (N., 2023)

d_α is a right-invariant and left-natural pseudo-metric, which is either non-degenerate or vanishes identically, such that

- if M is compact, (M, ξ) is strongly orderable if and only if d_α is non-degenerate,
- $\mathcal{O}_{\ll} = \mathcal{O}_{d_\alpha}$ for compact M,
- $d_\alpha \leq d_{SH}$.

Legendrian submanifolds

$L^n \subseteq M^{2n+1}$ is *Legendrian* if $TL \subseteq \xi$.

- Isotopies of Legendrian submanifolds are induced by ambient contact isotopies.
  $\Rightarrow \delta_\alpha^\pm$ induce distances $\delta_\alpha^{\pm,L}$ on the Legendrian isotopy class of L.
- We define $d_\alpha^L := \max\{\delta_\alpha^{+,L}, \delta_\alpha^{-,L}, 0\}$.
- For compact L, L is *strongly orderable* if there does not exists a positive loop of Legendrians starting at L.

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Main result for Legendrians

Theorem (N., 2023)

d^L_α is a right-invariant and left-natural pseudo-metric, such that

- if L is connected, d^L_α is either non-degenerate or vanishes identically,
- if L is compact and connected, L is strongly orderable if and only if d^L_α is non-degenerate,
- $\mathcal{O}^L_{\ll} = \mathcal{O}_{d^L_\alpha}$ for compact L,
- $d^L_\alpha \leq d_{SCH}$.

Universal covers

Remark

There are analogous results on the universal covers of $\mathrm{Cont}_0(M,\xi)$ and Legendrian isotopy classes, if we replace *positive loop* by *contractible positive loop* and *non-degenerate* by *the distance of two elements is positive if the underlying contactomorphisms/Legendrians are different*.

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Relation to the Reeb flow

Proposition

If (M, ξ) is strongly orderable,

$$\delta_\alpha^\pm(\phi_0, \phi_t^\alpha \phi_1) = \delta_\alpha^\pm(\phi_0, \phi_1) \pm t. \quad (8)$$

If L is strongly orderable,

$$\delta_\alpha^{\pm,L}(L_0, \phi_t^\alpha(L_1)) = \delta_\alpha^{\pm,L}(L_0, L_1) \pm t. \quad (9)$$

Remark

For τ_α, this was first shown by (Hedicke, 2021).

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Relation to the Reeb flow

REMARK

- (M, ξ) is strongly orderable if and only if $d_{SH}(Id_M, \phi_t^\alpha) = |t|$.
- L is strongly orderable if and only if $d_{SCH}(L, \phi_t^\alpha(L)) = |t|$.

THEOREM (N.,2021)

If L admits a loose chart of size $c > 0$ and the Reeb flow is complete, then $d_{SCH}(L, \phi_t^\alpha(L)) \leq c$.

COROLLARY

If L admits a loose chart of size $c > 0$ and the Reeb flow is complete, L admits a positive loop of energy less than $2c + \varepsilon$ ($\varepsilon > 0$).

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Spectrality

Remark

In the case that M is closed, or that L is closed and the Reeb flow is complete, the two main results were independently proven by (Allais-Arlove,2023).

They also showed that $\pm\delta_{\alpha}^{\pm,L}(L_0, L_1)$ takes values in the spectrum of actions of Reeb chords from L_0 to L_1.

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Thank you!

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