# Hamiltonian fragmentation in dimension four with applications to spectral estimators

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# History

Fragmentation $\phi = \phi_1 \circ \cdots \circ \phi_N$

Control on $\mathrm{Supp}(\phi_i)$: [Banyaga 97]

Control on $\mathrm{Supp}(\phi_i)$ and $N$: [Banyaga 97] for $C^1$-small Hamiltonians

Control on $\phi_i$: [Fathi 80] for $\mathbb{D}^2$

Control on $N$ and $\phi_i$: [Le Roux 09] for $\mathbb{D}^2$ for $C^0$-small Hamiltonians

Control on $N$ and $\phi_i$: [EPP 09] for all surfaces and for $C^0$-small Hamiltonians (Suggested to use theory of pseudo-holomorphic curves)

Control on $N$ and $\phi_i$:[A. 23] for $\mathbb{D}^2 \times \mathbb{D}^2$ for $C^0$-small Hamiltonians

# Pseudo-holomorphic curves

Theorem:([MS12]) Let $(X, \omega)$ be compact connected 4-dimensional satisfying:

1. No symp-emb 2-sphere with self-intersection $-1$.
2. $A, B \in H_2(X, \mathbb{Z})$ satisfying $A.B = 1,\ A.A = 0,\ B.B = 0$.
3. $A, B$ are represented by symp-emb 2-spheres.

Then: if $U, V \subset S^2$ open disks, $\iota : U \times S^2 \cup S^2 \times V \to X$ a given embedding satisfying:

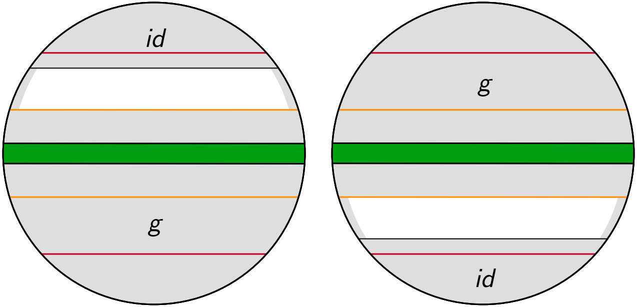
$$\iota^*\omega = a\pi_1^*\sigma + b\pi_2^*\sigma, \quad a = \int_A \omega,\ b = \int_B \omega$$

$$\iota_*([S^2 \times \{w\}]) = A, \quad \iota_*([\{z\} \times S^2]) = B$$

then, for any closed subsets $D \subset U$ and $C \subset V$ there is a symp-diff $\psi : S^2 \times S^2 \to X$ extending $\iota$ on $D \times S^2 \cup S^2 \times C$.

# Extension

Lemma: $X = S^2(a) \times S^2(b)$ and $M = \mathbb{D}^2(\frac{a}{2}) \times \mathbb{D}^2(\frac{b}{2})$. Let $g \in \mathrm{Ham}_M(X)$ that is $C^0$-small enough. Then, there exist a $\psi \in \mathrm{Ham}(X)$ compactly supported in $\mathbb{D}^2(\frac{a}{2}) \times D_3$ that restricts to $g$ on $\mathbb{D}^2(\frac{a}{2}) \times D_1$.

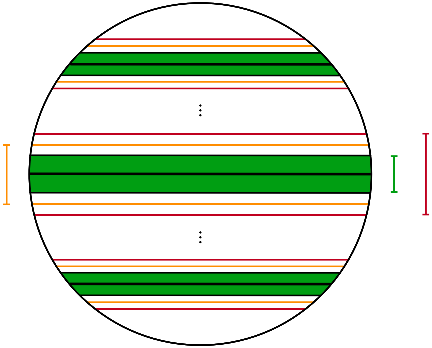



$\psi_1$ $\psi_2$

$$\psi = \psi_1 \circ \psi_2 \circ g^{-1}$$

# Fragmentation

Lemma: Let $M = \mathbb{D}^2(\frac{a}{2}) \times \mathbb{D}^2(\frac{b}{2})$. Then, $\forall \epsilon > 0$, $\exists\ \delta > 0$ and $N > 0$ s.t. for every $g \in \mathrm{Ham}_c(M)$ with $d_{C^0}(g, id) < \delta$ there are $g_i \in \mathrm{Ham}(M)$, $i = 1, \ldots, N$ each supported in a (union of disks) disk with area $< \epsilon$ and $g = g_1 \circ \cdots \circ g_N$.



# Hofer approximation by small supports

Theorem (A.): Let $X := S^2(a) \times S^2(b)$ and $M := \mathbb{D}^2(\frac{a}{2}) \times \mathbb{D}^2(\frac{b}{2})$, $B$ a proper disk in $S^2(b)$. Then, $\forall\ \epsilon > 0, \exists\ \delta > 0$ s.t.: for every $g \in \mathrm{Ham}_M(X)$ with $d_{C^0}(g, id) < \delta$ there exist $\psi \in \mathrm{Ham}_{\mathbb{D}^2(\frac{a}{2})\times B}(X)$ satisfying

$$d_H(g, \psi) < \epsilon.$$

Proof:

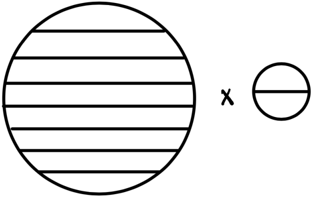
$$g = g_1 \circ \cdots \circ g_N,$$
$$g = (g_{i_1} \ldots g_{i_k}) \circ \cdots \circ (g_{i_{(N-1)/k+1}} \ldots g_{i_{N-1}}) \circ g_N$$
$$g = f_1 \circ \cdots \circ f_{(N-1)/k} \circ f_N$$

$$\psi = (h_1 f_1 h_1^{-1}) \circ \cdots \circ (h_{(N-1)/k} f_{(N-1)/k} h_{(N-1)/k}^{-1}) \circ (h_N f_N h_N^{-1})$$

# Application: PS-spectral estimators

Theorem (PS21): Let $M_a = S^2(1) \times S^2(a)$. There exists a map $c_{k,B} : C^\infty(M_a \times [0,1] \to \mathbb{R})$ satisfying:

- Hofer-Lipschitz
- Monotonicity
- Normalization
- Lagrangian control
- Independence of Hamiltonian
- Sub-additivity
- Calabi away from Lagrangians
- Controlled additivity



1. Hofer-Lipschitz: $|c_{k,B}(H) - c_{k,B}(G)| \leq \int_0^1 \max|H_t - G_t| dt$.
2. Lagrangian-control: If $H_t(.)|_{L^j_{k,B}} \equiv c_j(t)$ then,

$$c_{k,B}(H) = \frac{1}{k} \sum_{0 \leq j < k} \int_0^1 c_j(t) dt.$$

3. Independence of Hamiltonian: $c_{k,B}(H)$ only depends on $[\{\widetilde{\phi}_H^t\}]$ if $H$ is normalized.
4. Calabi property: If $\mathrm{supp}(H) \subset U \times [0,1]$ where $U \cap L_{k,B} = \emptyset$, then,

$$c_{k,B}(H) = -\frac{1}{\mathrm{vol}(M)} \mathrm{Cal}(H).$$

5. Controlled additivity: If $\mathrm{supp}(H) \subset U \times [0,1]$ where $U \cap L_{k,B} = \emptyset$, then, for all $G$:

$$c_{k,B}(\widetilde{\phi_H} \widetilde{\phi_G}) = c_{k,B}(\widetilde{\phi_H}) + c_{k,B}(\widetilde{\phi_G}).$$

# $C^0$-continuity of $\tau_{k,k',B,B'}$

Theorem 1 (A.23): The invariant $\tau_{k,k',B,B'} : \mathrm{Ham}(\mathbb{D}^2(c) \times \mathbb{D}^2(a)) \to \mathbb{R}$ defined by $c_{k,B} - c_{k',B'}$ is uniformly $C^0$-continuous for $0 < c < 1$ and small enough $a > 0$.

Theorem 2 (A.23): The complement of a Hofer ball in $\mathrm{Ham}_c(\mathbb{D}^2(c) \times \mathbb{D}^2(a))$ contains a $C^0$-open subset for all $0 < c < 1$ and rational $0 < a < 1$.

Theorem 3 (A.23): The group $\mathrm{Ham}_c(\mathbb{D}^2(c) \times \mathbb{D}^2(a))$ equipped with the Hofer distance admits an isometric embedding of $(C_c^\infty(0, b), d_{C^0})$ for parameters $0 < a, c < 1$ satisfying

$$b < \frac{1}{6}(1 - a), \ \frac{1}{2} + b < c < 1.$$

$\tau_{k,k',B,B'}$hank you