On the Hofer-Zehnder Conjecture for Semipositive Symplectic Manifolds

Joint work with Marcelo Atallah

Han Lou
University of Georgia

Symplectic Zoominar
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Outline

- Background
- key part : Upper bound of boundary depth
- Remarks

Background

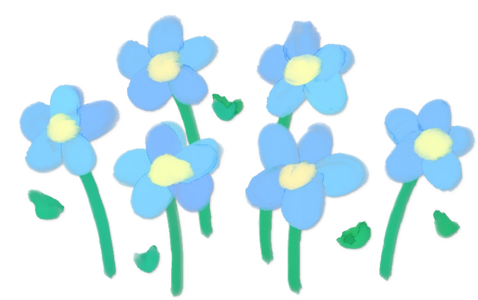
Semipositive Symplectic Manifold

- compact
- for every $A \in \pi_2(M)$

  $3-n \leq c_1(A) < 0$ implies $\omega(A) \leq 0$

Example

- S^2
- Calabi-Yau manifold
- Symplectic manifolds with dimension ≤ 6



Background

Hamiltonian $H : S^1 \times M \longrightarrow \mathbb{R}$ smooth

Hamiltonian vector field $\omega(X_H, \cdot) = dH$

k-periodic orbits ϕ_H^t flow of X_H with $\phi_H^0 = id$

x is a fixed point of ϕ_H^k

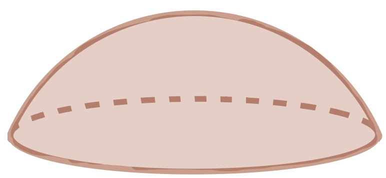
$x(t) = \phi_H^{kt}(x)$

Hamiltonian diffeomorphism ϕ_H^1

Background

Action Functional on Contractible Loops in M

$$\mathcal{A}_H([x, v]) = \int_0^1 H(t, x(t))dt - \int_{D^2} v^*\omega$$



$v : D^2 \longrightarrow M$

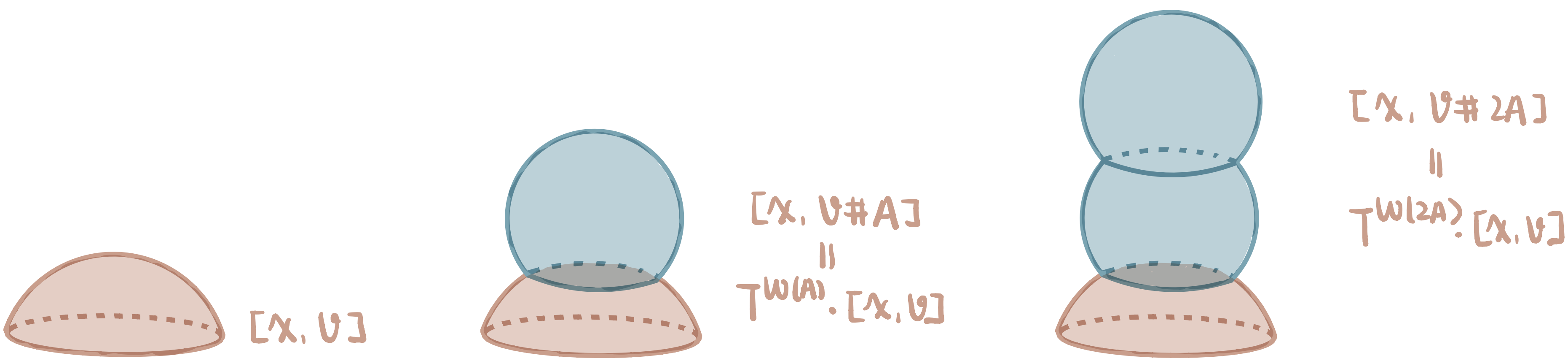
$v(e^{2\pi i t}) = x(t)$

Background

Floer homology is the Morse homology of the action functional $\mathcal{A}_H$

Coefficient $\Lambda_{\mathbb{k},\omega} = \{ \sum k_i T^{\omega(A_i)} \mid k_i \in \mathbb{k}$ field, $c_1(A_i) = 0$, finite negative powers$\}$

$\Lambda_{\mathbb{k},\omega}$-module structure $T^{\omega(A)} \cdot [x, v] = [x, v \# A]$



Extend coefficients to $\Lambda_{\mathbb{k},univ} = \{ \sum k_i T^{\lambda_i} \mid k_i \in \mathbb{k}$ field, $\lambda_i \uparrow \infty \}$

$$HF(H, \Lambda_{\mathbb{k},univ}) = HF(H, \Lambda_{\mathbb{k},\omega}) \otimes \Lambda_{\mathbb{k},univ}$$

Background

$$\Lambda^{0}_{\mathbb{K},\mathrm{univ}} \overset{\text{subring}}{\subseteq} \Lambda_{\mathbb{K},\mathrm{univ}}$$

$$\Lambda^{0}_{\mathbb{K},\mathrm{univ}} = \left\{ \sum k_i T^{\lambda_i} \mid \lambda_i \geq 0 \right\}$$

$$HF(H, \Lambda^{0}_{\mathbb{K},\mathrm{univ}}) = \left(\bigoplus \Lambda^{0}_{\mathbb{K},\mathrm{univ}} \right) \oplus \left(\frac{\Lambda^{0}_{\mathbb{K},\mathrm{univ}}}{(T^{\beta_1})} \oplus \cdots \oplus \frac{\Lambda^{0}_{\mathbb{K},\mathrm{univ}}}{(T^{\beta_k})} \right)$$

Bar lengths $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_k$

β_k: Usher's boundary depth

Background

Quantum Homology $QH(M, \Lambda_{\Bbbk,univ}) = H(M, \Bbbk) \otimes \Lambda_{\Bbbk,univ}$

with ring structure given by Gromov-Witten invariants

PSS Isomorphism (Piunikhin - Salamon - Schwarz)

$$HF(H, \Lambda_{\Bbbk,univ}) \cong QH(M, \Lambda_{\Bbbk,univ})$$

Motivation

J. Franks (1992, 1996)	For S^2, $\#\{1\text{-periodic orbits}\} > 2$ implies infinitely many periodic orbits.
Arnold conjecture	$\#\{1\text{-periodic orbits}\} \geq \dim(HF(M))$
Hofer-Zehnder conjecture (1994)	If "$>$", then there are infinitely many periodic orbits.
E. Shelukhin (2019)	Yes, when M is monotone and the even quantum homology is semisimple
&	
non-monotone blowup of $\mathbb{CP}^2$ $\cap$	
Atallah - Lou	Yes, when M is semipositive and the even quantum homology is semisimple
four points blowup of $\mathbb{CP}^2$ for certain size of exceptional divisors $\cup$ &	
S. Bai and G. Xu (2023)	Yes, for all toric manifolds

Ostrover–Tyomkin's 8-dimensional monotone toric manifold whose quantum homology is not semisimple

Main Result
Atallah – L.

Let (M, ω) be a closed semipositive symplectic manifold whose even quantum homology $QH_{ev}(M, \Lambda_{\mathbb{Q}, univ})$ is semisimple. Let ϕ be a nondegenerate Hamiltonian diffeomorphism such that

$$\infty > \#\{\text{contractible 1-periodic orbits}\} > \dim_{\mathbb{Q}} H(M, \mathbb{Q})$$

Then ϕ has infinitely many contractible periodic orbits.

Y. Sugimoto's definition of equivariant Floer homology

E. Shelukhin's algebraic argument

Assume finiteness

$$p \cdot \sum \beta_i(\phi, \Lambda_{\mathbb{Z}_p, univ}) \leq \sum \beta_i(\phi^p, \Lambda_{\mathbb{Z}_p, univ})$$

$\neq 0$

$\#\{\text{contractible 1-periodic orbits}\} > \dim_{\mathbb{Q}} H(M, \mathbb{Q})$

$\leq \#\text{bars} \cdot \text{boundary depth}$

$\leq \#\text{bars} \cdot C$

independent of p

Independent of p for p sufficiently large

Key Part

$QH_{ev}(M, \Lambda_{\mathbb{Q},univ})$ is semisimple. Then there is an upper bound of boundary depth of $HF(H; \Lambda_{\mathbb{Z}_p,univ})$ that is independent of p for sufficiently large p

$$QH_{ev}(M, \Lambda_{\mathbb{Q},univ}) = \bigoplus_{i=1}^{n} H_{2i}(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q},univ}$$

Semisimple : $QH_{ev}(M, \Lambda_{\mathbb{Q},univ}) \cong Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m$

Q_i is a field extension of $\Lambda_{\mathbb{Q},univ}$

$Q_i = e_i * QH_{ev}(M, \Lambda_{\mathbb{Q},univ})$

$e_i * e_i = e_i$, $e_i * e_j = 0$ Idempotents

$e_1 + \cdots + e_m = [M]$

$QH^{ev}(M, \overline{Frac(\Lambda_{\mathbb{Z},univ})})$ is semisimple

Idempotents $\bar{e}_i = \sum_{j=1}^{n} k_{ij} h_j$

$k_{ij} \in \overline{Frac(\Lambda_{\mathbb{Z},univ})}$

$h_j \in H_{2j}(M, \mathbb{Z})$

k_{ij} — $Frac(\Lambda_{\mathbb{Z},univ})(\alpha)$

$\downarrow$

$k_{ij}(x) + (f(x))$ — $\dfrac{Frac(\Lambda_{\mathbb{Z},univ})[x]}{(f(x))}$

$\downarrow$

$[k_{ij}(x)]_p + ([f(x)]_p)$ — $\dfrac{\Lambda_{\mathbb{Z}_p,univ}[x]}{([f(x)]_p)}$

$\downarrow$

$[k_{ij}(x)]_p + (g_1(x) \cdots g_k(x))$ — $\dfrac{\Lambda_{\mathbb{Z}_p,univ}[x]}{(g_1(x) \cdots g_k(x))}$

$\downarrow$

$[k_{ij}(x)]_p + (g_i(x))$ — $\dfrac{\Lambda_{\mathbb{Z}_p,univ}[x]}{(g_i(x))}$

$\downarrow$

$[k_{ij}]_p$ — $\overline{\Lambda_{\mathbb{Z}_p,univ}}$

$\{[\bar{e}_i]_p\}$ are idempotents in $QH^{ev}(M, \overline{\Lambda_{\mathbb{Z}_p,univ}})$

$QH^{ev}(M, \overline{\Lambda_{\mathbb{Z}_p,univ}})$ is semisimple and is generated by $\{[\bar{e}_i]_p\}$

The relation between $\{\bar{e}_i\}$ and $\{[\bar{e}_i]_p\}$ gives an upper bound of boundary depth of $HF(\phi, \overline{\Lambda_{\mathbb{Z}_p,univ}})$ that is independent of p

Remark on Semipositivity

P. Seidel 2014	$\mathbb{Z}_2$-equivariant fixed point Floer cohomology on Liouville domain.
E. Shelukhin and J. Zhao, 2019	Generalize P. Seidel's work to $\mathbb{Z}_p$.
E. Shelukhin, 2019	$\mathbb{Z}_p$-equivariant Hamiltonian Floer homology on monotone symplectic manifold.
Y. Sugimoto, 2021	$\mathbb{Z}_p$-equivariant Hamiltonian Floer homology on semipositive symplectic manifold.

If $\mathbb{Z}_p$-equivariant Hamiltonian Floer homology can be defined in more general setting, then it is possible to prove the Hofer-Zehnder conjecture in more general setting.

Remark on semisimple even quantum homology

The Conley Conjecture : For a broad class of closed symplectic manifolds, every Hamiltonian diffeomorphism has infinitely many simply periodic orbits.

Hold : Calabi-Yau manifolds , Negative monotone symplectic manifolds.

Fail : S^2, $\mathbb{C}P^n$, complex Grassmannians.

$$\begin{tikzcd} & \text{Conley Conjecture Fails} \arrow[dl, Leftrightarrow, "?"'] \arrow[dr, Rightarrow, "\text{Chance-McDuff Conjecture}"] & \\ \text{Even quantum homology is semisimple} \arrow[rr, Rightarrow] & & \exists \text{ nonvanishing Gromov-Witten invariants or a nontrivially deformed quantum product} \end{tikzcd}$$

THANK YOU