Subleading asymptotics of symplectic Weyl laws

Oliver Edtmair

UC Berkeley

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Overview

- Weyl laws
  - What can be said about the subleading asymptotics?
- Symplectic packing
  - How much volume can be covered by disjoint symplectic images of balls?
- Algebraic structure of transformation groups
  - What are the normal subgroups of $\overline{\mathrm{Ham}}(M)$?

Classical Weyl law

(M^n, g) compact Riemannian manifold, possibly with boundary
$0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots < \infty$ eigenvalues of $-\Delta_g$
$N(\lambda) :=$ number of eigenvalues less than λ

Theorem (Weyl 1911)

$N(\lambda) = (2\pi)^{-n}\omega_n \operatorname{vol}(M)\lambda^{n/2} + E(\lambda) \quad \text{with} \;\; E(\lambda) = o(\lambda^{n/2})$

Theorem (Levitan, Avakumovic, Seeley 50s)

$E(\lambda) = O(\lambda^{(n-1)/2})$

Remark: this is sharp for the round sphere

Theorem (Duistermaat-Guillemin, Ivrii 70s)

If the set of closed geodesics has measure zero, then
$E(\lambda) = -\frac{1}{4}(2\pi)^{1-n}\omega_{n-1} \operatorname{vol}(\partial X)\lambda^{(n-1)/2} + o(\lambda^{(n-1)/2})$

Remark: fails for round sphere

Embedded contact homology (ECH) Weyl law

$X \subset \mathbb{R}^4$ star-shaped domain $\rightsquigarrow$ ECH capacities

$$0 < c_1(X) \leq c_2(X) \leq \cdots < \infty$$

Spectrality property: For every k, we can find finitely many closed orbits $\gamma_i \subset \partial X$ such that $c_k(X) = \sum_i \mathcal{A}(\gamma_i)$

Theorem (Hutchings '10)

For all star-shaped domains $X \subset \mathbb{R}^4$ we have

$$c_k(X) = 2(\mathrm{vol}(X)k)^{1/2} + o(k^{1/2}) \qquad (k \to \infty).$$

Cristofaro-Gardiner-Hutchings-Ramos ('12): More general Weyl law for arbitrary contact 3-manifolds
Application: C^∞ closing lemma for 3D Reeb flows (Irie '15)

Periodic Floer homology (PFH) Weyl law

Closed surface (Σ, ω) of area A, Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times \Sigma \to \mathbb{R}$
$\rightsquigarrow$ PFH spectral invariants $c_1(H), c_2(H), \dots \in \mathbb{R}$

Theorem (CG-Prasad-Zhang, E.-Hutchings 2021)

For all Hamiltonians H we have

$$c_d(H) = dA^{-1} \int_{\mathbb{R}/\mathbb{Z}\times\Sigma} H dt \wedge \omega + o(d) \qquad (d \to \infty).$$

- Similar statement for area preserving diffeomorphisms
- Related Weyl law for link spectral invariants (CG-Humilière-Mak-Seyfaddini-Smith, Shelukhin-Polterovich+Buhovsky)

Applications: C^∞ closing lemma, Simplicity conjecture (CG-Humilière-Seyfaddini),...

Subleading asymptotics

For $X \subset \mathbb{R}^4$ star-shaped write $c_k(X) = 2(\mathrm{vol}(X)k)^{1/2} + e_k(X)$

Theorem (Hutchings '19)

We have $e_k(X) = O(k^{1/4})$ as $k \to \infty$.

- Slightly weaker bounds for general contact 3-manifolds by CG-Savale and Sun

Question: In all known examples $e_k(X) = O(1)$. Always true?

Theorem (Hutchings '19)

If X is a strictly convex or concave toric domain then

$$\lim_{k\to\infty} e_k(X) = -\frac{1}{2}Ru(X). \tag{1}$$

Counterexample: $Ru(B(a)) = 2a$ but
$\liminf_{k\to\infty} e_k(B(a)) = -3a/2$ $\qquad$ $\limsup_{k\to\infty} e_k(B(a)) = -a/2$
Question: Is (1) true for generic X?

Relationship with symplectic packing

ECH Weyl law $\quad c_k(X) = 2(\mathrm{vol}(X)k)^{1/2} + o(k^{1/2})$
Sketch of proof:
Step 1: true for ball ("direct" computation)
Step 2: true for disjoint unions of balls

$$c_k(\coprod_i X_i) = \max_{\sum_i k_i = k} \sum_i c_{k_i}(X_i)$$

Step 3: Let X be star-shaped, $\varepsilon > 0$ arbitrary. There exists disjoint union $B = \coprod_i B_i$ of finitely many balls such that

- $B \overset{s}{\hookrightarrow} X$
- $\mathrm{vol}(B) \geq \mathrm{vol}(X) - \varepsilon$

$\Rightarrow \quad c_k(X) \geq c_k(B) \geq 2((\mathrm{vol}(X) - \varepsilon)k)^{1/2} + o(k^{1/2})$
Step 4: For the reverse inequality consider a big ball $C \supset X$ and fill $C \setminus X$ by small balls

Relationship with symplectic packing

For (disjoint unions of) balls we have $e_k = O(1)$.
Question: Why does this proof not show $e_k(X) = O(1)$ for all star-shaped X?

Let B_n denote the disjoint union of n equal balls with total volume $\mathrm{vol}(B_n) = 1$. We have

$$\limsup_{k\to\infty} e_k(B_n) \longrightarrow -\infty \quad (n \to \infty)$$

If we can pack the full volume of X and $C \setminus X$ by finitely many balls, we get $e_k(X) = O(1)$.

Symplectic packing stability

Let (M, ω) be a symplectic manifold of finite volume. Define the kth ball packing number by

$$p_k(M) := \sup_{a>0} \frac{k \cdot \mathrm{vol}(B(a))}{\mathrm{vol}(M)}$$

where the supremum is taken over all a such that $\coprod_{i=1}^{k} B(a) \overset{s}{\hookrightarrow} M$.

Theorem (McDuff-Polterovich '94)

We have $\lim_{k\to\infty} p_k(M) = 1$.

Theorem (Biran '99)

Suppose that (M, ω) is a closed, rational symplectic 4-manifold. Then there exists k_0 such that for all $k \geq k_0$ we have $p_k(M) = 1$.

Definition: We say (M, ω) has *packing stability* if the assertion of the above theorem holds.

Symplectic packing stability

Question (Cieliebak, Hofer, Latschev, Schlenk '07)

Which finite volume (M, ω) have packing stability?

- Buse-Hind '13: closed rational symplectic manifolds and ellipsoids in any dimension
- Buse-Hind-Opshtein '16: closed symplectic 4-manifolds, 4-dimensional polydisks
- CG-Holm-Mandini-Pires '21: 4-dimensional rational convex toric domains

Theorem (CG-Hind '23)

There exists a bounded open subset $U \subset \mathbb{R}^4$ diffeomorphic to the open ball for which packing stability fails.

Remark: U is not symplectomorphic to the interior of a compact symplectic manifold with piecewise smooth boundary.

Symplectic packing stability

Question

Does packing stability hold for compact symplectic manifolds with (piecewise) smooth boundary?

Almost nothing known:

- Packing stability holds for ellipsoids, polydisks or more generally rational convex toric domains, but these domains can be approximated by divisor complements in closed symplectic manifolds.
- The space of symplectic structures on a closed manifold is finite dimensional (Moser stability).
- The space of symplectic structures on a manifold with boundary is at least as complex as the set of conjugacy classes in Symp (the characteristic foliation could admit a Poincaré section).

Simplicity

Theorem (Banyaga '78)

Let (M, ω) be a closed symplectic manifold. Then $\mathrm{Ham}(M)$ is a simple group.

Corollary

Let $\alpha \in \mathrm{Ham}(M)$ be not the identity. Then for every $\varphi \in \mathrm{Ham}(M)$, there exist N and $\psi_1, \ldots, \psi_N \in \mathrm{Ham}(M)$ such that

$$\varphi = \prod_{i=1}^{N} \psi_i \alpha^{\pm 1} \psi_i^{-1}.$$

Main idea:

Packing manifold with dynamically complicated boundary by simple pieces (balls)	$\longleftrightarrow$	Decomposing dynamically complicated diffeomorphism into conjugates of a simpler one

Results in progress

Theorem (in progress)

Packing stability holds for every compact, connected, symplectic 4-manifold with smooth boundary.

Corollary

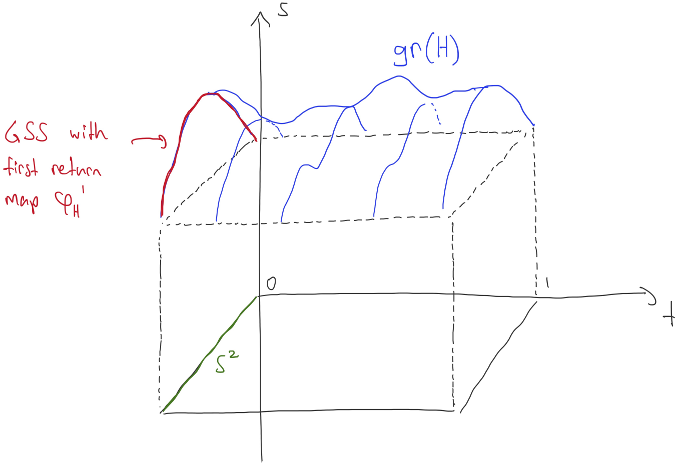
- (ECH) For all star-shaped domains $X \subset \mathbb{R}^4$ we have
$$c_k(X) = 2(\mathrm{vol}(X))^{1/2} + O(1).$$
- (PFH) For all Hamiltonians $H : \mathbb{R}/\mathbb{Z} \times \Sigma \to \mathbb{R}$ we have
$$c_d(H) = dA^{-1} \int_{\mathbb{R}/\mathbb{Z}\times\Sigma} Hdt \wedge \omega + O(1).$$

Remark: For a general (Y^3, ξ): If $e_k = O(1)$ for one single contact form, then $e_k = O(1)$ for all contact forms.

A toy case - setup

Equip $M := \mathbb{R}_s \times (\mathbb{R}/\mathbb{Z})_t \times S^2$ with $\Omega := ds \wedge dt + \omega$. Given a Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$, define the **truncated subgraph**

$$\mathrm{gr}_-(H) := \{(s,t,p) \in M \mid 0 \leq s \leq H(t,p)\} .$$



A toy case - theorem

Theorem (E. '23)

For every Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$ and every sufficiently large constant $C \geq 0$, the truncated subgraph $\text{gr}_-(H + C)$ can be fully packed by finitely many balls.

Corollary

Let $H : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$ be a Hamiltonian on (Σ, ω). Then

$$c_d(H) = dA^{-1} \int_{\mathbb{R}/\mathbb{Z} \times \Sigma} H dt \wedge \omega + O(1).$$

A toy case - sketch of proof

Given: $H : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$
Goal: full ball packing of $\mathrm{gr}_-(H + C)$

- Let $R : S^2 \to \mathbb{R}$ be scaled height function such that

$$\varphi_R^1 = \text{half rotation of } S^2.$$

- Banyaga: there exist $\psi_1, \ldots, \psi_N \in \mathrm{Ham}(S^2)$ such that

$$\varphi_H^1 = \prod_i \psi_i \circ \varphi_R^1 \circ \psi_i^{-1}.$$

- Define $G : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$ by

$$G(t, z) := N \cdot R(\psi_i^{-1}(z)) \quad \text{for } (i-1)/N \leq t \leq i/N.$$

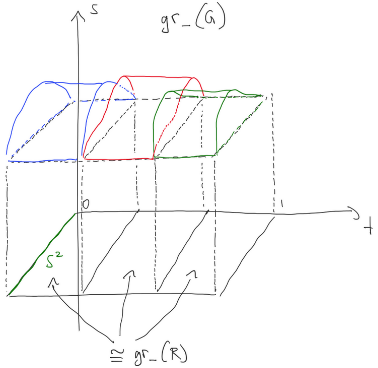
- Have $\varphi_H^1 = \varphi_G^1$. Can arrange equality in $\widetilde{\mathrm{Ham}}(S^2)$.

A toy case - sketch of proof (continued)

- Have $\varphi_H^1 = \varphi_G^1$. Can arrange equality in $\widetilde{\mathrm{Ham}}(S^2)$.
- After shift $H \rightsquigarrow H + C$ and $R \rightsquigarrow R + D$ for suitable $C, D > 0$:

$$\mathrm{gr}\_(H) \overset{s}{\cong} \mathrm{gr}\_(G).$$

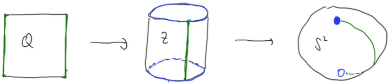
- $\mathrm{gr}\_(G)$ admits packing by N copies of $\mathrm{gr}\_(R)$.
- Suffices to pack $\mathrm{gr}\_(R)$ by balls.

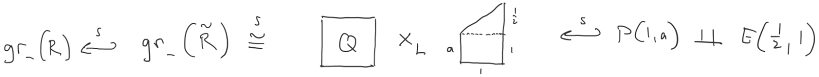


A toy case - sketch of proof (continued)

- Suffices to pack $\mathrm{gr}\_(R)$ by balls.
- Set $I := [0,1]$ $\quad Q := I^2$ $\quad Z := I \times \mathbb{R}/\mathbb{Z}$.
- We cut $\mathrm{gr}\_(R)$ as follows:

$$\widetilde{R} : I \times Q \to I \times Z \to I \times S^2 \xrightarrow{R} \mathbb{R} \qquad \widetilde{R}(t,x,y) = a + x/2$$





- Polydisks and ellipsoids can be packed by balls. $\square$

The simplicity conjecture

Definition (Hamiltonian homeomorphisms $\overline{\mathrm{Ham}}(M,\omega)$)

$\varphi \in \mathrm{Homeo}(M)$ is a *Hamiltonian homeomorphism* if it is a uniform limit of Hamiltonian diffeomorphisms.

Definition (Hameomorphisms $\mathrm{Hameo}(M,\omega)$)

$\varphi \in \mathrm{Homeo}(M)$ is called a *Hameomorphism* if there exist $H \in C^0([0,1] \times M)$ and $(H_k)_k \subset C^\infty([0,1] \times M)$ such that

- $\|H - H_k\|_{(1,\infty)} \to 0$
- $d_{C^0}(\varphi, \varphi^1_{H_k}) \to 0$.

Theorem (CG-Humilière-Seyfaddini + Mak-Smith)

Let Σ be a closed surface. Then $\mathrm{Hameo}(\Sigma,\omega)$ is a proper normal subgroup of $\overline{\mathrm{Ham}}(\Sigma,\omega)$.

Theorem (CG-Humilière-Mak-Seyfaddini-Smith + Mak-Trifa)

$\mathrm{Hameo}(\Sigma,\omega)$ is not simple either.

C^0 non-simplicity and failure of packing stability

Theorem (CG-Hind '23)

There exists a bounded open subset $U \subset \mathbb{R}^4$ diffeomorphic to the open ball for which packing stability fails.

- Smooth Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$:

simplicity of $\mathrm{Ham}(S^2)$	$\Longrightarrow$	full ball packing of $\mathrm{gr}_-(H)$	$\Longrightarrow$	$c_d(H)$ have $O(1)$ subleading asymptotics

- Continuous Hamiltonian $H : \mathbb{R}/\mathbb{Z} \times S^2 \to \mathbb{R}$ generating $\varphi \in \mathrm{Hameo}(S^2)$:

non-simplicity of $\mathrm{Hameo}(S^2)$	$\overset{?}{\Longleftarrow}$	no full ball packing of $\mathrm{gr}_-(H)$	$\Longleftarrow$	failure of $O(1)$ subleading asymptotics for $c_d(H)$

Thank you!

A conjecture

For finite volume (M^{2n}, ω) and $x > 0$ define

$$v(M, \omega; x) := \sup_{a>0} \frac{\mathrm{vol}(a \cdot E(1, x, \dots, x))}{\mathrm{vol}(M)}$$

where the supremum is taken over all $a > 0$ such that
$a \cdot E(1, x, \dots, x) \overset{s}{\hookrightarrow} M$.

Conjecture

For every compact, connected, symplectic manifold (M^{2n}, ω) with smooth boundary, there exists $x_0 > 0$ such that, for all $x \geq x_0$, we have $v(M, \omega; x) = 1$.

Remark: This is known for closed rational symplectic manifolds and ellipsoids (Buse-Hind).

What made the toy case easy?

1. ability to shift H up
2. existence of a closed global surface of section
3. existence of a rotation on S^2

Dealing with the absence of 1:

Theorem (Quantitative perfectness)

The commutator length is bounded on some C^∞ open neighbourhood of the identity in $\mathrm{Ham}(M)$.