C^0-stability of topological entropy for Reeb flows in dimension 3

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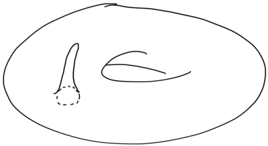
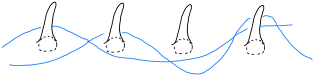
Introduction

(T^2, g): 2-torus with a bump



Implies the existence of various types of geodesics on (T^2, g) (Bialy-Polterovich, Polterovich, Bangert, Bolotin-Rabinowitz, ...)

- Rmk.: oscillation of geodesics around bump.



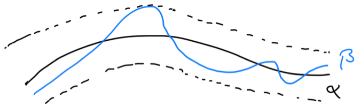
- Rmk.: “oscillation behaviour” is robust.

C^0 distance between contact forms

(Y, ξ): closed co-oriented contact 3-manifold.
$\mathcal{R}(Y, \xi)$: the set of contact forms α on (Y, ξ),
$(\alpha \wedge d\alpha > 0, \ker \alpha = \xi)$.
Every $\alpha \in \mathcal{R}(Y, \xi)$ defines its **Reeb** vector field R_α on (Y, ξ) by:

$$d\alpha(R_\alpha, \cdot) = 0$$
$$\alpha(R_\alpha) = 1.$$

For $\alpha, \beta \in \mathcal{R}(Y, \xi)$, write $\alpha = f_{\alpha,\beta}\beta$ for a smooth function $f_{\alpha,\beta} : Y \to (0, +\infty)$.



The C^0**-distance between** α **and** β is

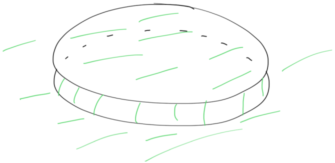
$$d_{C^0}(\alpha, \beta) = \max|\log f_{\alpha,\beta}| \quad (= \max|\log f_{\beta,\alpha}|).$$

Related distances: Contact Banach-Mazur distance
(Ostrover-Polterovich, Stojisavljevic-Zhang, Usher, Rosen-Zhang),

C^0-distance between Riemannian metrics

S closed surface. $\mathrm{Met}(S)$: space of Riemannian metrics on S.
C^0-**distance** on $\mathrm{Met}(S)$:

$$\overline{d}_{C^0}(g,g') = \inf\left\{\epsilon > 0 \mid e^{-\epsilon}|v|_{g'} \leq |v|_g \leq e^{\epsilon}|v|_{g'}\right\}.$$



Remark: Under the usual assignment
$\Phi : \mathrm{Met}(S) \to \mathcal{R}(Y, \xi_{\mathrm{geo}}), g \mapsto \alpha_g$,
we have $\Phi^* d_{C^0} = \overline{d}_{C^0}$.

Topological entropy

M closed manifold, ϕ flow on M.
d: auxiliary distance function on M.

- Given $T, \delta > 0$, a subset $S \subset M$ is said to be (T, δ)**-separated** if, for all points $p, q \in S$ with $p \neq q$, we have

$$\max_{t \in [0,T]} \{d(\phi^t(p), \phi^t(q))\} > \delta.$$

- $n_\phi^\delta(T)$: maximal cardinality of a (T, δ)-separated set.
- δ**-entropy:**

$$h_\delta(\phi) := \limsup_{T \to +\infty} \frac{\log(n_\phi^\delta(T))}{T},$$

- **topological entropy:**

$$h_{\mathrm{top}}(\phi) := \lim_{\delta \to 0} h_\delta(\phi).$$

Main results

Theorem 1 (Alves-Dahinden-M.-Pirnapasov, '23)
Let (Y, ξ) be a closed contact 3-manifold. The topological entropy

$$\begin{aligned} h_{\text{top}} : \mathcal{R}(Y,\xi) &\to [0, +\infty), \\ \alpha &\mapsto h_{\text{top}}(\phi_\alpha^t) \end{aligned}$$

is lower semi-continuous with respect to d_{C^0} on a C^∞-open and dense set.

- Here: We say h_{top} is "lower semi-continuous on" $U \subset \mathcal{R}(Y,\xi)$ if every $\alpha \in U$ is a point of lower semi-continuity ($\liminf_{\alpha' \to \alpha} h_{\text{top}}(\phi_{\alpha'}) \geq h_{\text{top}}(\phi_\alpha)$).

Main results

Given a closed surface S, denote be $\mathrm{Met}_{\mathrm{nd}}(S) \subset \mathrm{Met}(S)$ the set of non-degenerate Riemannian metrics on S.

Theorem 2 (ADMP, '23)

The topological entropy

$$\begin{aligned} h_{\mathrm{top}} : \mathrm{Met}(S) &\to [0, +\infty), \\ g &\mapsto h_{\mathrm{top}}(\phi_g^t) \end{aligned}$$

is lower semi-continuous with respect to $\overline{d}_{C^0}$ on $\mathrm{Met}_{\mathrm{nd}}(S)$.

Corollaries (of the proofs)

Let $(Y,\xi) = (S^3, \xi_{\mathrm{tight}})$. A contact form on (Y,ξ) is called **right-handed** if all their trajectories link positively.
In right-handed flows every periodic orbit is binding of a global surface of section. (Ghys)

Example: Pinched metrics on S^2

(Florio-Hryniewicz): If $g \in \mathrm{Met}(\mathrm{S}^2)$ is δ-pinched with $\delta > 0.7225$, then the geodesic flow ϕ_g lifts to a right-handed flow on S^3.
(g is **δ-pinched** if $\delta \leq K_{\mathrm{min}}/K_{\mathrm{max}}$.)

Corollary 1

Let $(Y,\xi) = (S^3, \xi_{\mathrm{tight}})$. The topological entropy

$$h_{\mathrm{top}} : \mathcal{R}(Y,\xi) \to [0, +\infty)$$

is lower semi-continuous with respect to d_{C^0} on the set of right-handed $\alpha \in \mathcal{R}(Y,\xi)$.

Corollaries

Our techniques together with an earlier result (Alves-Dahinden-M.-Merlin, '21) yield also

Corollary 2

For **every** $g \in \mathrm{Met}(T^2)$ with $h_{\mathrm{top}}(\phi_g) > 0$ there is a $\overline{d}_{C^0}$-neighbourhood U of g such that

$$h_{\mathrm{top}}(\phi_{g'}) > 0 \quad (\forall g' \in U).$$

Conjectures

Conjecture 1

Given a closed co-oriented contact 3-manifold (Y, ξ), the entropy functional

$$h_{\mathrm{top}} : \mathcal{R}(Y, \xi) \to [0, +\infty)$$

is lower semi-continuous with respect to d_{C^0}.

Conjecture 2

Given a closed orientable surface S, the entropy functional

$$h_{\mathrm{top}} : \mathrm{Met}(S) \to [0, +\infty)$$

is lower semi-continuous with respect to $\overline{d}_{C^0}$.

Clearly, Conjecture 1 implies Conjecture 2.

Further context

- topological entropy of Reeb flows/Contactomorphisms (studied by Alves, Dahinden, Frauenfelder, Macarini, M., Schlenk, ...).
- Alves-Pirnapasov: Forcing and Reeb flows.

Robustness/lower semi-continuity

- question by L. Polterovich
- robustness features of h_{top} wrt. d_{C^0}: Dahinden, Alves-Dahinden-M.-Merlin.
- Alves-M. ('21): lower semi-continuity of h_{top} wrt. the Hofer metric on $\mathrm{Ham}(S, \omega)$, $\dim(S) = 2$.
- Hutchings ('23): lower semi-continuity of h_{top} wrt. Hofer metric for area preserving diffeos.

Context: Barcode entropy

- Barcode entropy: First introduced and studied by Çineli-Ginzburg-Gürel ('21). Further results by Mazzucchelli-Ginzburg-Gürel, Fender-Lee-Sohn, ...
- Mazzucchelli-Ginzburg-Gürel ('22): $\hbar(\phi_g) = h_{\mathrm{top}}(\phi_g)$ for $g \in \mathrm{Met}(S)$, $\dim(S) = 2$.
- (MGG + Theorem 2): The barcode entropy

$$\hbar : \mathrm{Met}(S) \mapsto [0, +\infty),$$
$$g \mapsto \hbar(\phi_g),$$

is lower semi-continuous with respect to $\overline{d}_{C^0}$ on $\mathrm{Met}_{\mathrm{nd}}(S)$.

Hypertight in the complement of a link

Let α be a contact form, $\mathcal{L}$ a link of closed Reeb orbits of α.

Definition

We say that α is **hypertight in the complement of** $\mathcal{L}$ if any disk map into *M* whose boundary parametrizes a closed Reeb orbit has an interior intersection with $\mathcal{L}$.

Example: If $\mathcal{L}$ bounds a global surface of section that is not a disk, then α_0 is hypertight in the complement of $\mathcal{L}$.

Definition

A homotopy class ρ of loops in $M \setminus \mathcal{L}$ is a **proper link class** if

- no loop in ρ is contained in a small tubular neighbourhood of a component of $\mathcal{L}$.
- Momin: if α hypertight in the complement of $\mathcal{L}$, ρ proper link class, all orbits in ρ non-degenerate $\rightarrow$ cylindrical contact homology in class ρ ($\mathrm{CH}^{\rho}_{\mathcal{L}}(\alpha)$) is well defined.

Homotopical growth

Let $\alpha_0 \in \mathcal{R}(Y,\xi)$, $\mathcal{L}$ link of Reeb orbits, α_0 hypertight in the complement of $\mathcal{L}$.

- **homotopical growth** $\Gamma_{\mathcal{L}}(\alpha_0)$: exponential growth rate of proper link classes ρ with $\mathrm{CH}^{\rho}_{\mathcal{L}}(\alpha_0) \neq 0$ (Alves-Pirnapasov)
- $\Gamma_{\mathcal{L}}(\alpha_0) \leq h_{\mathrm{top}}(\phi_{\alpha_0})$ (Alves-Pirnapasov)

It will be useful to restrict to classes ρ with one periodic orbit:

- $\Omega_{\alpha_0}(\mathcal{L})$: set of proper link classes ρ that carry **exactly one** orbit
- $\Omega^T_{\alpha_0}(\mathcal{L})$: set of $\rho \in \Omega_{\alpha_0}(\mathcal{L})$ with orbit of period $\leq T$
- Clearly: $\limsup_{T\to+\infty} \frac{\log\left(\#\Omega^T_{\alpha_0}(\mathcal{L}^{\alpha_0}_{\epsilon})\right)}{T} \leq \Gamma_{\mathcal{L}}(\alpha_0)$ (if orbits are non-degenerate)

Approximation of h_{top} by $\Gamma_{\mathcal{L}}$

Theorem (M. '23)

Let $\alpha_0 \in \mathcal{R}(Y, \xi)$. Assume that $h_{\mathrm{top}}(\phi_{\alpha_0}) > 0$. For any $0 < \epsilon < h_{\mathrm{top}}(\phi_{\alpha_0})$, there exists a link $\mathcal{L}_\epsilon^{\alpha_0}$ defined by hyperbolic periodic orbits of ϕ_{α_0} such that

$$\limsup_{T \to +\infty} \frac{\log\left(\#\Omega_{\alpha_0}^T(\mathcal{L}_\epsilon^{\alpha_0})\right)}{T} > h_{\mathrm{top}}(\phi_{\alpha_0}) - \epsilon.$$

As a consequence:

$$h_{\mathrm{top}}(\phi_{\alpha_0}) = \sup_{\mathcal{L}} \Gamma_{\mathcal{L}},$$

whenever $h_{\mathrm{top}}(\phi_{\alpha_0}) > 0$, and $\Gamma_{\mathcal{L}_0}$ can be defined **for some** $\mathcal{L}_0$.

Remarks on the approximation theorem

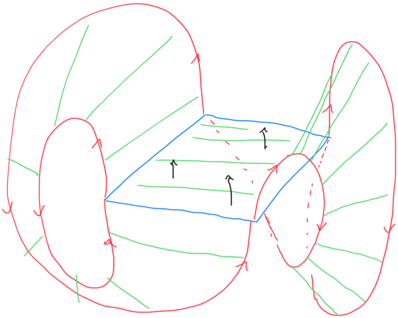
- Katok: for surface diffeos φ: Approximation of $h_{\mathrm{top}}(\varphi)$ by h_{top} of horseshoes.
- Versions for 3D flows: Lima-Sarig (countable Markov shifts "coding" the dynamics of the return map to a Poincaré section), Lian-Young.

The link $\mathcal{L}_\epsilon^{\alpha_0}$ in the approximation theorem

- link $\mathcal{L} = \mathcal{L}_\epsilon^{\alpha_0}$ in our result can be written as

$$\mathcal{L} = \mathcal{L}^0 \cup \bigcup_{k=1}^{n} \mathcal{L}^k,$$

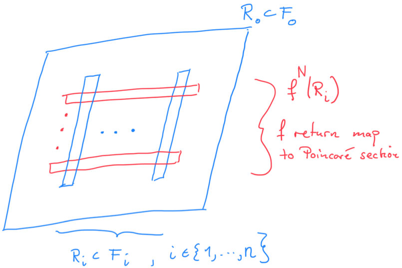
where each $\mathcal{L}^k$, $k \in \{0, \ldots, n\}$, bounds a pair-of-pants F_k (Fried surface) transversal to the flow.



- $F_1, \dots, F_n$ are pairwise disjoint, and all intersect F_0.
- There is an injective map

$$\phi : (\mathcal{S}/ \sim) \to \Omega_{\alpha_0}(\mathcal{L}),$$

  where $\mathcal{S} = \bigcup_{m \in \mathbb{N}} \{1, \dots, n\}^m$, and $\underline{i} \sim \underline{j}$ if $\sigma(\underline{i}) = \underline{j}$ for a cyclic permutation σ.
- The unique periodic orbit γ in the class $\phi([(i_1, \dots, i_m)])$ intersects the surfaces $F_1, \dots, F_n$ in the order $\dots, F_{i_1}, \dots, F_{i_m}, \dots$.
  Each time γ intersects some F_k, $k \in \{1, \dots, n\}$, it intersects also F_0.



Stability

Theorem (ADMP)

Let α_0 be **non-degenerate**, $\mathcal{L}_0$ a link of Reeb orbits of α_0 such that α_0 is **hypertight in the complement** of $\mathcal{L}_0$. Assume that

$$a := \limsup_{T \to +\infty} \frac{\log\left(\#\Omega^T_{\alpha_0}(\mathcal{L}_0)\right)}{T} > 0.$$

Let $\epsilon > 0$. Then, $\exists \delta > 0$ such that for every non-degenerate contact form α with $d_{C^0}(\alpha, \alpha_0) < \delta$, there is a link $\mathcal{L}(\alpha)$ of Reeb orbits of α satisfying:

$$\limsup_{T \to +\infty} \frac{\log\left(\#\Lambda^T_{\alpha}(\mathcal{L}(\alpha))\right)}{T} > a - \epsilon.$$

$\Lambda^T_\alpha(\mathcal{L}(\alpha))$: set of homotopy classes of loops in $M \setminus \mathcal{L}(\alpha)$ with a periodic orbit of period $\leq T$.

Remarks on stability

- We can choose δ (size of the allowed d_{C^0}-neighbourhood) independent of the choice of contact form α_0 outside a neighbourhood of $\mathcal{L}_0$ (as long as assumptions hold).
- The growth rate $\Gamma_{\mathcal{L}(\alpha)}(\alpha)$ might not be well-defined.
- We do not know if the new link $\mathcal{L}(\alpha)$ in the statement can be chosen to be isotopic to $\mathcal{L}_0$. Moreover, it might have more, or less components.

Proof of Theorem 1

Theorem 1

The topological entropy $h_{\text{top}} : \mathcal{R}(Y, \xi) \to [0, +\infty)$ is lower semi-continuous with respect to d_{C^0} on a C^∞-open and dense set.

on C^∞-dense set:

- Assume α_0 is non-degenerate and there is a global surface of section (gss) for the flow; such forms are C^∞ dense (Colin-Dehornoy-Hryniewicz-Rechtman, Contreras-Mazzucchelli).
- $\mathcal{L}_0$: union of the binding of the gss and the link from approximation result:
  $\limsup_{T \to +\infty} \frac{\log\left(\#\Omega^T_{\alpha_0}(\mathcal{L}_0)\right)}{T} \geq h_{\text{top}}(\phi_{\alpha_0}) - \epsilon/2.$
- Stability result: for some $\delta > 0$, and all non-degenerate α with $d_{C^0}(\alpha, \alpha_0) < \delta$ there is $\mathcal{L}(\alpha)$ with:
  $\limsup_{T \to +\infty} \frac{\log\left(\#\Lambda^T_{\alpha}(\mathcal{L}(\alpha))\right)}{T} \geq h_{\text{top}}(\phi_{\alpha_0}) - \epsilon.$
- $\Rightarrow h_{\text{top}}(\phi_\alpha) \geq h_{\text{top}}(\alpha_0) - \epsilon$, extends to degenerate α (Newhouse)

Proof of Theorem 2

- If g is non-degenerate, we can choose a surface of section with binding $\mathcal{L}_b$ such that all periodic orbits in $M \setminus \mathcal{L}_b$ are non-contractible in $M \setminus \mathcal{L}_b$ (Contreras-Knieper-Mazzucchelli-Schulz).
- for contractible binding components the two lifts of the underlying geodesic are linked (Pirnapasov).

Stability: Setting

α_0 non-degenerate, $\mathcal{L}_0$ link of Reeb orbits of α_0 such that α_0 is hypertight in the complement of $\mathcal{L}_0$,

$$a := \limsup_{T \to +\infty} \frac{\log\left(\#\Omega^T_{\alpha_0}(\mathcal{L}_0)\right)}{T} > 0.$$

α sufficiently C^0 close to α_0.

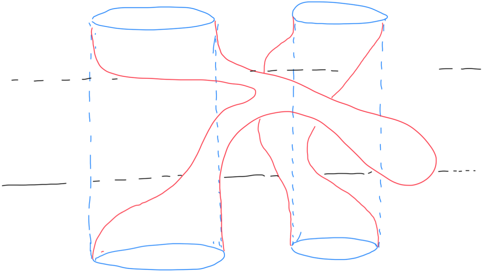
For any $R > 0$ we consider exact symplectic cobordism W^R equipped with compatible almost complex structure J^R such that

- in $[-R, R] \times Y$, W^R coincides with the symplectisation $(\mathbb{R} \times Y, d(e^r\alpha))$ of α, J^R is cylindrical.
- outside $[-R-1, R+1] \times Y$, W^R coincides with symplectisation of $C^+\alpha_0$, $C^-\alpha_0$ respectively, $C^- < 1 < C^+$, J^R cylindrical.

Step 1

There exists

- a collection of pairwise disjoint finite energy holomorphic cylinders V^R in W^R, positively and negatively asymptotic to $\mathcal{L}_0$,
- an ambient isotopy from $\mathbb{R} \times \mathcal{L}_0$ to V^R (asymptotic to identity).



Important: $C^{\pm}$ sufficiently close to 1 (compactness of relevant moduli spaces), positivity of intersections.

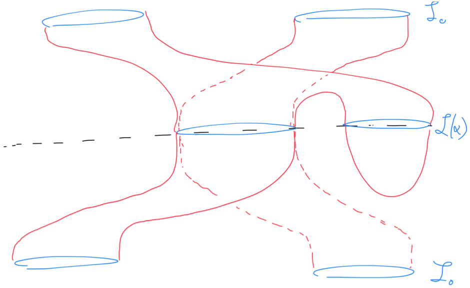
Step 2

for any $\rho \in \Omega_{\mathcal{L}_0}(\alpha_0)$, there exists a finite energy holomorphic cylinder u_ρ in $W^R \setminus V^R$ positively/negatively asymptotic to the unique orbit γ_ρ in ρ.

Important: Ambient isotopy from Step 1, assumptions on $\mathcal{L}_0$ and ρ, and positivity of intersections.

Step 3

Let $R \to +\infty$: top and bottom level of the buildings coming from V^R and u_ρ in the SFT-limit are the cobordism levels.
$V^{R,-}$, u_ρ^-: bottom levels.

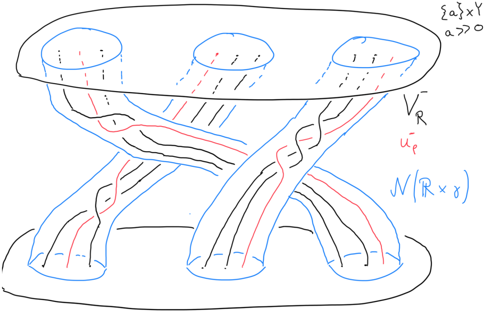


$\mathcal{L}(\alpha)$: positive asymptotic orbits of the components of $V^{R,-}$.

Step 4

Let γ be periodic orbit of α.
Growth of the number of $\rho \in \Omega^T_{\alpha_0}$ such that u^-_ρ is positively asymptotic to γ is at most quadratic in T.

(symptotic behaviour of holomorphic curves asymptotic to $\gamma \in \mathcal{L}(\alpha)$ (Siefring), ρ proper link class).



Conclusion: $\limsup_{T\to+\infty} \frac{\log\left(\#\Lambda^T_\alpha(\mathcal{L}(\alpha))\right)}{T} \geq a - \epsilon.$

Thanks for listening!