The coarse distance from dynamically convex to convex

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Symplectic convexity

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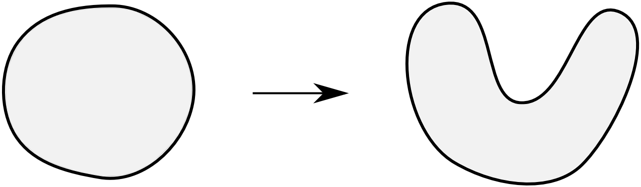

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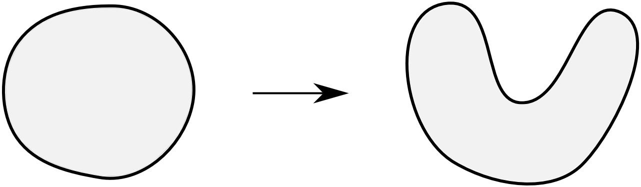

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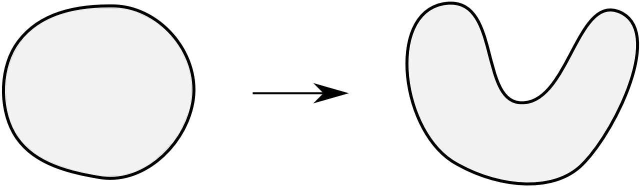

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Definition (symplectically convex domains)

$\mathcal{C}_4 = \{$*domains of* $\mathbb{R}^4$ *which are symplectomorphic to a convex domain*$\}$

Dynamical convexity

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Theorem (Chaidez and Edtmair, 2020)

There exists dynamically convex domains of $\mathbb{R}^4$ which are not symplectically convex.

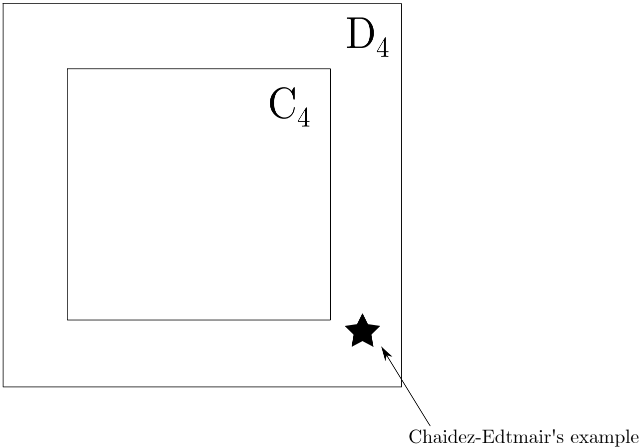

Figure – Relations between $\mathcal{C}_4$ and $\mathcal{D}_4$.

$\mathcal{C}_4 = \{$symplectically convex domains of $\mathbb{R}^4\}$
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Theorem (D., Gutt, Ramos and Zhang, 2023)

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Remark

These are the first examples of dynamically convex domains which are not symplectically convex without referring to Chaidez-Edtmair's criterion.

Coarse symplectic Banach-Mazur distance

(Ostrover-Polterovich)
For $U, V \subset \mathbb{R}^4$ star-shaped domains, let

$$d_c(U, V) = \inf\left\{\log \lambda \geq 0 \,\middle|\, \frac{1}{\lambda}U \hookrightarrow V \hookrightarrow \lambda U\right\}$$

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Open problem

If $d_c(U, V) = 0$, U is symplectomorphic to V ?

A "new" symplectic convexity criterion

Theorem (John, 1948)

Let U be a convex domain of $\mathbb{R}^4$, then there exists an ellipsoid $E \subset \mathbb{R}^4$ such that

$$E \subset U \subset o + 4 \cdot (E - o)$$

where o is the center of E.

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Proposition (Symplectic John's ellipsoid theorem)

Let U be a symplectically convex domain of $\mathbb{R}^4$, then

$$d_c(U, \mathcal{E}_4) := \inf_{E \in \mathcal{E}_4} d_c(U, E) \leq \log 2$$

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Every toric domain can be written as $X_\Omega = \mu^{-1}(\Omega)$ *where* $\Omega \subset (\mathbb{R}_{\geq 0})^2$ *and*

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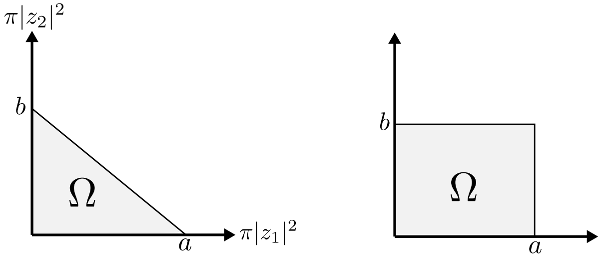

Figure – An ellipsoid $E(a, b)$ and a polydisc $P(a, b)$

Monotone toric domains

Definition

A ***monotone toric*** *domain is a compact toric domain with a smooth boundary such that for every* $\mu \in \partial_+\Omega = \partial\Omega \cap (\mathbb{R}_{>0})^2$ *the outward normal vector at* μ *has non-negative components.*

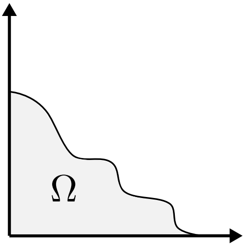

Figure – A monotone toric domain

Proposition (Gutt, Hutchings and Ramos, 2020)

Let $\mathcal{M}_4 = \{$*Monotone toric domains of* $\mathbb{R}^4\} \subset \mathcal{T}_4$

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Theorem (D., Gutt and Zhang, 2021)

There exists a family of monotone toric domains of $\mathbb{R}^4$ *which are not symplectically convex.*

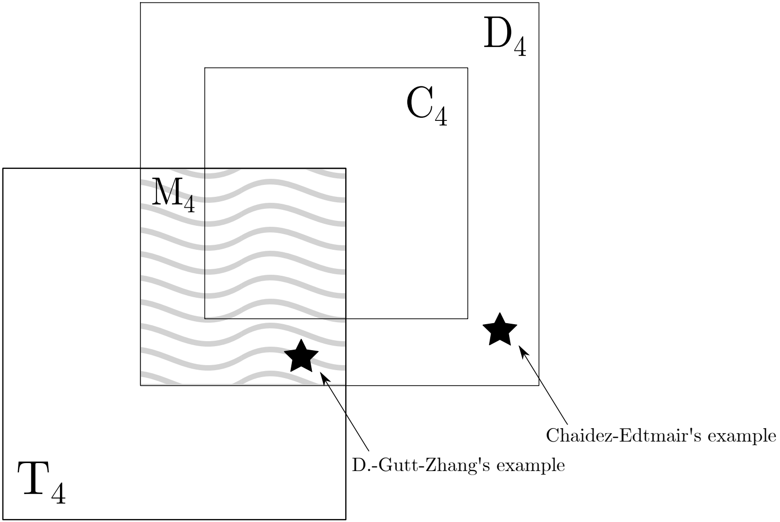

Figure – Relations between $\mathcal{T}_4$, $\mathcal{C}_4$, $\mathcal{M}_4$, and $\mathcal{D}_4$.

The L^p ball

For $p \in (0, 1]$, let

$$X_{\Omega_p} := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \left(\pi |z_1|^2\right)^p + \left(\pi |z_2|^2\right)^p < 1 \right\}.$$

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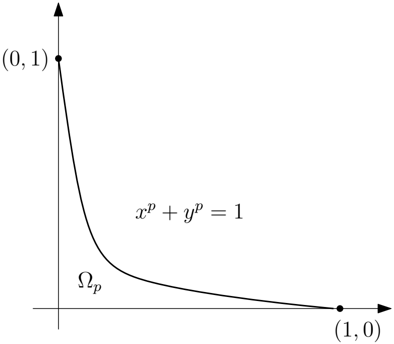

Figure – X_{Ω_p} for $p \in (0,1]$.

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Theorem (D., Gutt, Ramos and Zhang, 2023)

For toric domain X_{Ω_p} and $p < \frac{1}{5}$, we have :

$$d_c(X_{\Omega_p}, \mathcal{E}_4) \geq \frac{1}{8} \log \left(\frac{g(p)}{1 + \log 4 + \log g(p)} \right) \tag{1}$$

where $g(p) = 2^{\frac{2}{p}-2} \mathrm{Vol}_{\mathbb{R}^4}(X_{\Omega_p})$.

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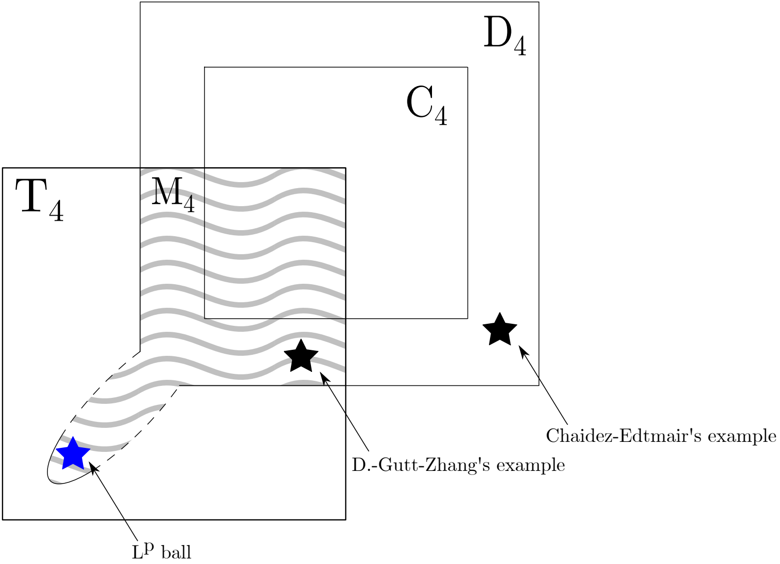
Corollary

In particular, when p satisfies the condition that

$$\frac{g(p)}{1 + \log 4 + \log g(p)} > 2^8, \tag{2}$$

then X_{Ω_p} is dynamically convex but not symplectically convex.


D_4
C_4
T_4
M_4
Chaidez-Edtmair's example
D.-Gutt-Zhang's example
L^p ball

Tools for the proof

Corollary's proof

Triangular inequality+symplectic John :

$$d_c(X_{\Omega_p}, \mathcal{E}_4) \leq d_c(X_{\Omega_p}, \mathcal{C}_4) + \sup_{U \in \mathcal{C}_4} d_c(U, \mathcal{E}_4)$$

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$$c_k^{ECH}(X_{\Omega_p}) = c_k^{ECH}\left(\bigsqcup_{i=1}^{k} B(w_i)\right)$$

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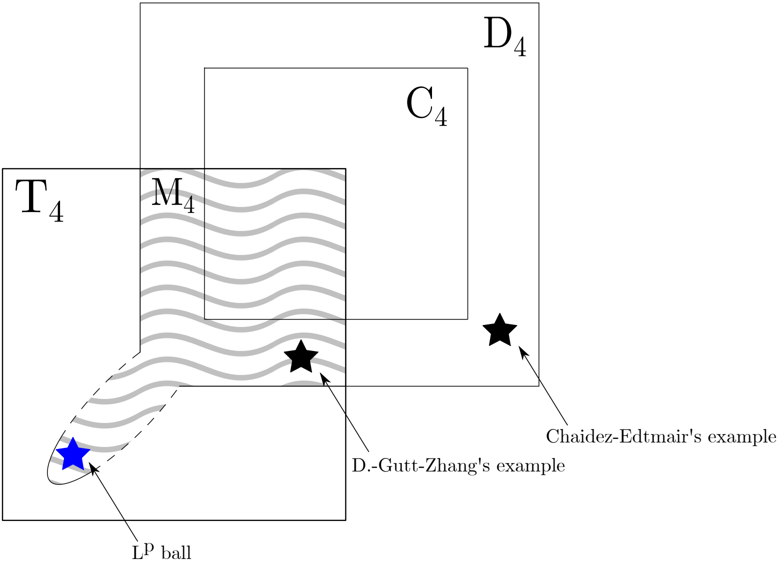
Lemma by Hutchings :

$$c_k^{ECH}\left(\bigsqcup_{i=1}^{k} B(w_i)\right) \leq 2\sqrt{k \cdot \text{vol}\left(\bigsqcup_{i=1}^{k} B(w_i)\right)}$$

Thank you !

Overview



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