The coarse distance from dynamically convex to convex

Julien Dardennes (joint work with J.Gutt, V.Ramos and J.Zhang)
Symplectic convexity

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**Theorem (Hofer, Wysocki and Zehnder, 1998)**

\[ C_4 \subset D_4. \]
Dynamical convexity

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$$\mathcal{D}_4 = \{\text{domains of } \mathbb{R}^4 \text{ which are dynamically convex } \}$$

Theorem (Hofer, Wysocki and Zehnder, 1998)

$$\mathcal{C}_4 \subset \mathcal{D}_4.$$  

Question : $$\mathcal{C}_4 = \mathcal{D}_4$$?
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\[ C_4 \subset D_4. \]

**Question:** \[ C_4 = D_4? \]

**Theorem (Chaidez and Edtmair, 2020)**

*There exists dynamically convex domains of \( \mathbb{R}^4 \) which are not symplectically convex.*
Figure – Relations between $\mathcal{C}_4$ and $\mathcal{D}_4$.

$\mathcal{C}_4 = \{\text{symplectically convex domains of } \mathbb{R}^4\}$

$\mathcal{D}_4 = \{\text{dynamically convex domains of } \mathbb{R}^4\}$
Main result

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Theorem (D., Gutt, Ramos and Zhang, 2023)

Dynamically convex domains are arbitrarily far from symplectically convex domains with respect to the coarse symplectic Banach-Mazur distance.
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Remark

These are the first examples of dynamically convex domains which are not symplectically convex without referring to Chaidez-Edtmair’s criterion.
Coarse symplectic Banach-Mazur distance

(Ostrover-Polterovich)
For $U, V \subset \mathbb{R}^4$ star-shaped domains, let

$$d_c(U, V) = \inf \left\{ \log \lambda \geq 0 \mid \frac{1}{\lambda} U \leftrightarrow V \leftrightarrow \lambda U \right\}$$

Open problem
If $d_c(U, V) = 0$, is $U$ symplectomorphic to $V$?
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A "new" symplectic convexity criterion

**Theorem (John, 1948)**

Let $U$ be a convex domain of $\mathbb{R}^4$, then there exists an ellipsoid $E \subset \mathbb{R}^4$ such that

$$E \subset U \subset o + 4 \cdot (E - o)$$

where $o$ is the center of $E$. 

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where $o$ is the center of $E$.

Proposition (Symplectic John’s ellipsoid theorem)
Let $U$ be a symplectically convex domain of $\mathbb{R}^4$, then

$$d_c(U, \mathcal{E}_4) := \inf_{E \in \mathcal{E}_4} d_c(U, E) \leq \log 2$$
Toric domains

A toric domain $X \subset \mathbb{C}^2 \cong \mathbb{R}^4$ is a domain that is invariant under the $\mathbb{T}^2$-action.

Figure – An ellipsoid $E(a, b)$ and a polydisc $P(a, b)$.
Toric domains

A toric domain \( X \subset \mathbb{C}^2 \cong \mathbb{R}^4 \) is a domain that is invariant under the \( T^2 \)-action.

**Proposition**

*Every toric domain can be written as* \( X_\Omega = \mu^{-1}(\Omega) \) *where* \( \Omega \subset (\mathbb{R}_{\geq 0})^2 \) *and*

\[
\mu : (z_1, z_2) \in \mathbb{C}^2 \mapsto \pi(|z_1|^2, |z_2|^2) \in (\mathbb{R}_{\geq 0})^2
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*Figure – An ellipsoid $E(a, b)$ and a polydisc $P(a, b)$*
Monotone toric domains

Definition

A **monotone toric** domain is a compact toric domain with a smooth boundary such that for every $\mu \in \partial_+ \Omega = \partial \Omega \cap (\mathbb{R}_{>0})^2$ the outward normal vector at $\mu$ has non-negative components.

Figure – A monotone toric domain
Proposition (Gutt, Hutchings and Ramos, 2020)

Let $\mathcal{M}_4 = \{Monotone \ toric \ domains \ of \ \mathbb{R}^4\} \subset \mathcal{T}_4$

$\mathcal{M}_4 = \mathcal{D}_4 \cap \mathcal{T}_4$
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Theorem (Gutt, Hutchings and Ramos, 2020)

All normalized symplectic capacities agree on $\mathcal{M}_4$.

Theorem (D., Gutt and Zhang, 2021)

There exists a family of monotone toric domains of $\mathbb{R}^4$ which are not symplectically convex.
Figure – Relations between $T_4$, $C_4$, $M_4$, and $D_4$. 
The $L^p$ ball

For $p \in (0, 1]$, let

$$X_{\Omega_p} := \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid \left( \pi |z_1|^2 \right)^p + \left( \pi |z_2|^2 \right)^p < 1 \right\}.$$
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Main result

Let $d_c(X_{\Omega_p}, \mathcal{E}_4) = \inf_{E \in \mathcal{E}_4} d_c(X_{\Omega_p}, E)$
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**Theorem (D., Gutt, Ramos and Zhang, 2023)**

For toric domain $X_{\Omega_p}$ and $p < \frac{1}{5}$, we have:

$$d_c(X_{\Omega_p}, \mathcal{E}_4) \geq \frac{1}{8} \log \left( \frac{g(p)}{1 + \log 4 + \log g(p)} \right)$$

(1)

where $g(p) = 2^{\frac{2}{p} - 2} \text{Vol}_{\mathbb{R}^4}(X_{\Omega_p})$.

**Corollary**

In particular, when $p$ satisfies the condition that $g(p) > 2^2 8^{-\frac{2}{p} + 2}$, (2) then $X_{\Omega_p}$ is dynamically convex but not symplectically convex.
Main result

Let \( d_c(X_{\Omega_p}, \mathcal{E}_4) = \inf_{E \in \mathcal{E}_4} d_c(X_{\Omega_p}, E) \)

**Theorem (D., Gutt, Ramos and Zhang, 2023)**

*For toric domain \( X_{\Omega_p} \) and \( p < \frac{1}{5} \), we have:*

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d_c(X_{\Omega_p}, \mathcal{E}_4) \geq \frac{1}{8} \log \left( \frac{g(p)}{1 + \log 4 + \log g(p)} \right)
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*where \( g(p) = 2^\frac{2}{p-2} \text{Vol}_{\mathbb{R}^4}(X_{\Omega_p}) \).*

**Corollary**

*In particular, when \( p \) satisfies the condition that*

\[
\frac{g(p)}{1 + \log 4 + \log g(p)} > 2^8,
\]

(2)

*then \( X_{\Omega_p} \) is dynamically convex but not symplectically convex.*
Tools for the proof

**Corollary’s proof**

Triangular inequality + symplectic John:

\[ d_c(X_{\Omega_p}, \mathcal{E}_4) \leq d_c(X_{\Omega_p}, \mathcal{C}_4) + \sup_{U \in \mathcal{C}_4} d_c(U, \mathcal{E}_4) \]
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Theorem’s proof: ECH capacities
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Theorem’s proof: ECH capacities
Weight decomposition:

\[ c^E_{ECH}(X_{\Omega p}) = c^E_{ECH} \left( \bigsqcup_{i=1}^{k} B(w_i) \right) \]
Tools for the proof

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Theorem’s proof: ECH capacities
Weight decomposition:

\[ c_k^{ECH}(X_{\Omega_p}) = c_k^{ECH} \left( \bigsqcup_{i=1}^{k} B(w_i) \right) \]

Lemma by Hutchings:

\[ c_k^{ECH} \left( \bigsqcup_{i=1}^{k} B(w_i) \right) \leq 2 \sqrt{k \cdot \text{vol} \left( \bigsqcup_{i=1}^{k} B(w_i) \right)} \]
Thank you!
Overview

D4
C4
Chaidez-Edtmair's example
T4 M4
D.-Gutt-Zhang's example
Lp ball

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