Complex projective spaces via surface groups representations

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Surface group representations

S : oriented surface

G : Lie group

$\rightsquigarrow \quad \chi(S,G) = \mathrm{Hom}(\pi_1 S, G) / G$

character variety

(1) [Atiyah–Bott 83', Goldman 84', ...]

Symplectic/Poisson structure on $\mathcal{X}(S,G)$

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symplectic / Poisson structure on $\mathcal{X}(S,G)$

(2) $\mathrm{Mod}(S) \curvearrowright \mathcal{X}(S,G)$

mapping class group of S

symplectic!

Compact examples

(1) G compact $\Rightarrow$ $\mathcal{X}(S, G)$ compact

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(2) what if G is not compact?

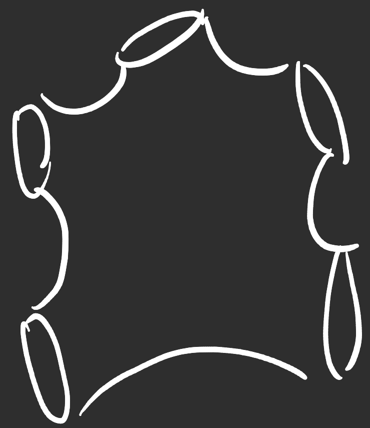
Totally elliptic representations

$\rho : \pi_1 S \longrightarrow G$ is totally elliptic if

$$\rho\left(\begin{array}{c}\text{simple}\\ \text{closed}\\ \text{curves}\end{array}\right) \subseteq \text{elliptic} \subseteq G$$

philosophy : totally elliptic $\rightsquigarrow$ compactness

Examples [Benedetto–Goldman 99', <u>Deroin–Tholozan</u> 20', Tholozan–Toulisse 21', ...]

$\pi_1\Big($  $\Big) \longrightarrow PSL_2\mathbb{R}$

punctured sphere

or any Hermitian Lie groups

$$\chi(\Sigma_n, PSL_2\mathbb{R})$$

$$\chi_{\alpha}(\Sigma_n, PSL_2\mathbb{R})$$

angles

$\alpha_1, \dots, \alpha_n \in (0, 2\pi)$

$\rho : \pi_1 \Sigma_n \longrightarrow PSL\,\mathbb{R}$

$\gamma_i \longmapsto$ elliptic of angle α_i

compact component of DT representations

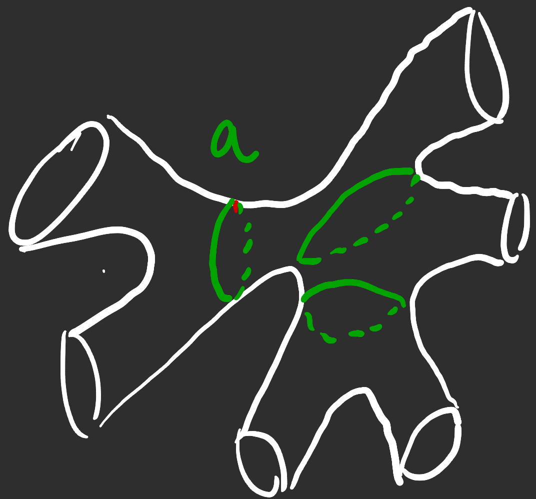
$$\chi_{\alpha}^{DT}(\Sigma_n, PSL_2\mathbb{R})$$

angles $\alpha_1, \dots, \alpha_n \in (0, 2\pi)$

$\rho : \pi_1 \Sigma_n \longrightarrow PSL\,\mathbb{R}$

$\partial_i \longmapsto$ elliptic of angle α_i

Hamiltonian torus action

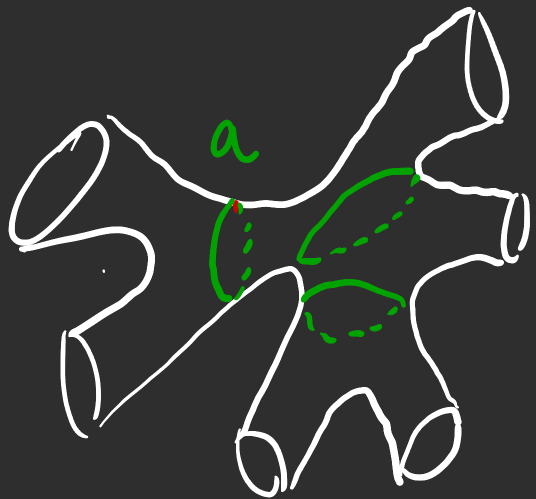


$\rightsquigarrow \vartheta_a : \chi_{\alpha}^{DT}(\Sigma_{n}, PSL_2\mathbb{R}) \longrightarrow (0, 2\pi)$

$\rho \longmapsto$ angle of $\rho(a)$

fact :

Hamiltonian torus action



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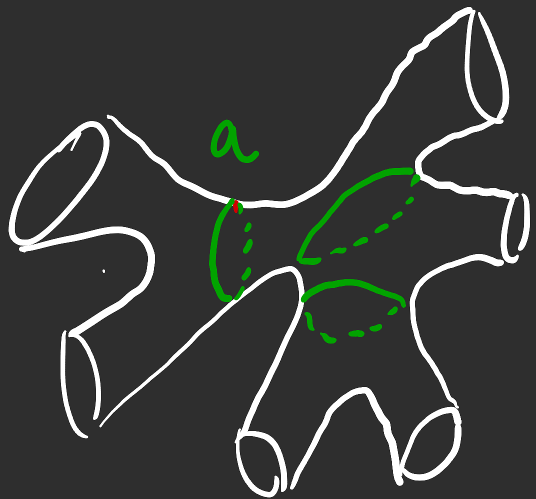
Hamiltonian torus action



$\rightsquigarrow \vartheta_a : \chi^{DT}_{\alpha}(\Sigma_n, PSL_2\mathbb{R}) \longrightarrow (0, 2\pi)$

$\rho \longmapsto$ angle of $\rho(a)$

<u>fact</u> :
* $\Phi^t_{\vartheta_a}$ is π-periodic
* $a \cap b = \emptyset \implies \Phi^t_{\vartheta_a}, \Phi^t_{\vartheta_b}$ commute

maximal number of disjoint s.c.c. on Σ_n $= \quad n-3 \quad = \frac{1}{2} \dim \mathcal{X}^{DT}_{\alpha}(\Sigma_n, PSL_2\mathbb{R})$

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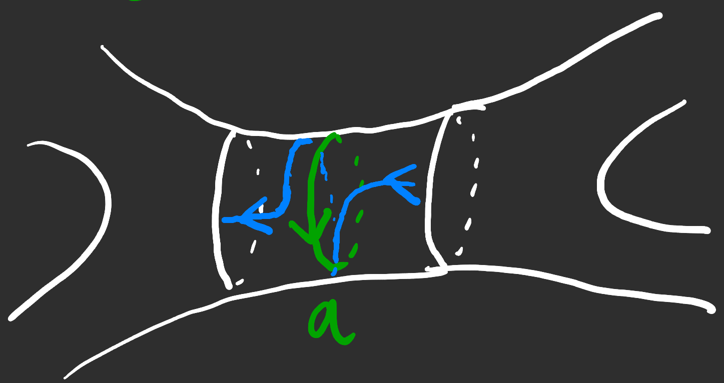
Delzant $\implies$ $\chi^{DT}_{\alpha}(\Sigma_n, PSL_2\mathbb{R}) \cong \mathbb{C}P^{n-3}$

$\uparrow$ Deroin–Tholozan

What new?

$$\tau_a = \Phi_{\vartheta_a}^{t=\vartheta_a/2}$$

Dehn twist



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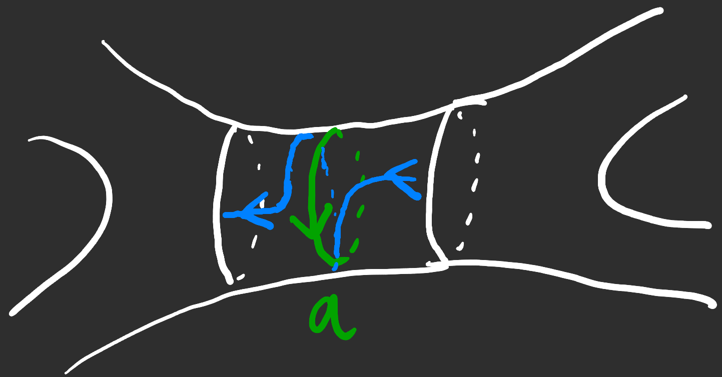

$$\mathbb{CP}^{n-3}$$

$$\cong$$

$$\mathrm{Mod}(\Sigma_n) < \mathrm{Ham}\left(\mathcal{X}^{DT}_{\alpha}(\Sigma_n, PSL_2\mathbb{R})\right)$$

Ongoing work w/ V. Humilière

Entov-Polterovich quasimorphism : $\mathrm{Mod}(\Sigma_n) \xrightarrow{\mu} \mathbb{R}$

$$\wedge$$

$$\mathrm{Ham}(\chi_\alpha^{DT}(\Sigma_n, PSL_2\mathbb{R}))$$

Ongoing work w/ V. Humilière

Entov-Polterovich quasimorphism : $\mathrm{Mod}(\Sigma_n) \xrightarrow{\mu} \mathbb{R}$

$$\mathrm{Ham}\left(\chi_{\alpha}^{DT}(\Sigma_n, PSL_2\mathbb{R})\right)$$

<u>THM</u> (Humilière-M. 23')

k = # punctures on one side of a

$$\mu(\tau_a) = C(\alpha, n)\cdot (k-1)(n-k-1)$$