Deformation inequivalent symplectic structures
and Donaldson’s four-six question

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Definition
Two closed symplectic manifolds \((X_1, \omega_1)\) and \((X_2, \omega_2)\) are deformation equivalent if there exists a diffeomorphism \(\phi : X_1 \to X_2\) such that \(\phi^* \omega_2\) is connected to \(\omega_1\) via a path of symplectic forms.

- No assumptions on cohomology classes.
- No assumptions on diffeomorphisms.
The “four-six” question

Question (Donaldson)
Given two closed homeomorphic symplectic 4-manifolds \((X_1, \omega_1)\) and \((X_2, \omega_2)\), are they diffeomorphic if and only if the product manifolds

\[(X_1 \times S^2, \omega_1 \oplus \omega_{\text{std}}) \simeq (X_2 \times S^2, \omega_2 \oplus \omega_{\text{std}})\]

Call this “stabilization”.

Theorem (Wall, ’64)

Two closed simply-connected homeomorphic 4-manifolds are h-cobordant.

Theorem (Smale, ’62)

Let \(n \geq 5\). Then two closed simply-connected \(n\)-manifolds are h-cobordant implies that they are diffeomorphic.
History of the question

- **Ruan ’94**: There exist homeomorphic but not diffeomorphic Kähler surfaces such that their stabilizations are not deformation equivalent. This is given by $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$ and the Barlow surface.

- **Ruan-Tian ’97**: Stated “Stabilizing Conjecture” when restricted to simply-connected 4-manifolds. Shown for rational elliptic surfaces.

- **Ionel-Parker ’99**: Also shown for $E(n)$ using different methods, but still used GW invariants.

- **Smith ’00**: Given $n \geq 2$, constructed $n$ symplectic forms on a simply-connected 4-manifold whose $c_1$s have different divisibilities. $\implies$ Donaldson’s question cannot be replaced by $\mathbb{T}^2$. 
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Main results

Theorem (Hirschi-W, ’23)

There exist infinitely many pairs of closed symplectic 4-manifolds $(X_1, \omega_1)$ and $(X_2, \omega_2)$ that are diffeomorphic, but their stabilizations $(X_1 \times S^2, \omega_1 \oplus \omega_{std}) \not\simeq (X_2 \times S^2, \omega_2 \oplus \omega_{std})$.

- In particular, we answer the forward direction of Donaldson’s four-six question in the negative.
- We give two “types” of examples, given by Smith and McMullen-Taubes.
Main results

Theorem (Hirschi-W, '23)

The examples above remain deformation inequivalent when stabilized with arbitrarily many copies of $(S^2, \omega_{\text{std}})$. 
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Proof strategy

- Find examples of a fixed smooth 4-manifold \( X \) that admits deformation inequivalent symplectic forms \( \omega_1 \) and \( \omega_2 \).
- Show that \( (X \times S^2, \omega_1 \oplus \omega_{\text{std}}) \not\approx (X \times S^2, \omega_2 \oplus \omega_{\text{std}}) \).

Invariant: (The orbit under Diff of) the first Chern class associated to a symplectic form.

Goal
Show that \( c_1(X \times S^2, \omega_1 \oplus \omega_{\text{std}}) \) and \( c_1(X \times S^2, \omega_2 \oplus \omega_{\text{std}}) \) lie in different orbits of \( \text{Diff}(X \times S^2) \).
More motivations for our proof

- Before us, there are already examples of a closed smooth 4-manifold $X$ admitting symplectic structures whose $c_1$s lie in different orbits of $\text{Diff}(X)$. In fact, we are using such examples.
- How to go from 4 to 6?
  - Done for diffeomorphisms of $X \times S^2$ that “split”.
  - Want to constrain how an arbitrary diffeomorphism of $X \times S^2$ can act on $H^2(X \times S^2)$.
Cohomology equivalences

Definition
Given $X$ and $Y$, let $G_{X,Y}$ denote the set of cohomology equivalences $\psi$ of $X \times Y$ such that
- $\psi^*$ maps $H^2(X; \mathbb{Z})$ to itself; and
- $\text{pr}_1 \psi(\cdot, y)$ is a cohomology equivalence for each $y \in Y$.

Definition
Let $\widetilde{G}_{X,Y} \subset G_{X,Y}$ be the $H$-group of homotopy equivalences.

Both types of our examples have their diffeomorphisms satisfying one of these algebraic conditions.
Proof steps

- Find a smooth 4-manifold $X$ with symplectic forms $\omega_1$ and $\omega_2$ such that $c_1(\omega_1)$ and $c_2(\omega_2)$ lie in different orbits of $\text{Diff}(X)$ cohomology (resp. homotopy) equivalences - this is stronger!

- Show that if $c_1(\omega_1)$ and $c_2(\omega_2)$ lie in different orbits of cohomology (resp. homotopy) equivalences, then $c_1(\omega_1 \oplus \omega_{\text{std}})$ and $c_1(\omega_2 \oplus \omega_{\text{std}})$ lie in different orbits of $G_{X,S^2}$ (resp. $\overline{G_{X,S^2}}$).

- Show that any diffeomorphism of $X \times S^2$ lies in $G_{X,S^2}$ (resp. $\overline{G_{X,S^2}}$).
Proposition

Let $X$ be a closed, smooth manifold with two symplectic forms $\omega_1$ and $\omega_2$. Suppose $c_1(\omega_1)$ and $c_1(\omega_2)$ are in different orbits of actions of cohomology (resp. homotopy) equivalences of $X$ on $H^2(X; \mathbb{Z})$. Then $c_1(\omega_1 \oplus \omega_{\text{std}})$ and $c_1(\omega_2 \oplus \omega_{\text{std}})$ lie in different orbits of action of $G_{X, S^2}$ (resp. $\tilde{G}_{X, S^2}$).
Proof of the algebraic “sufficient” condition

Suppose \( \exists \psi \in G_{X,S^2} \) (resp. \( \widetilde{G}_{X,S^2} \)), such that

\[
\psi^* c_1(\omega_2 \oplus \omega_{\text{std}}) = c_1(\omega_1 \oplus \omega_{\text{std}}). \tag{1}
\]

Then for \( h := \text{PD}[\text{pt}] = \text{AD}[S^2] = c_1 \) of the hyperplane line bundle, we have that \( \psi^* h = h + \alpha \) for some \( \alpha \in H^2(X) \). Also, \( \psi^*(h^2) = 0 \), implying

\[
(h + \alpha)^2 = h^2 + 2\alpha h + \alpha^2 = 0.
\]

So \( 2\alpha = 0 \), since \( H^*(X \times S^2; \mathbb{Z}) \cong H^*(X)[h]/h^2 \). Now, by (1) and \( c_1(\mathbb{C}P^1, \omega_{\text{std}}) = 2h \), we have that

\[
c_1(\omega_1) + 2h = \psi^* c_1(\omega_2) + 2\psi^* h = \psi^* c_1(\omega_2) + 2h + 2\alpha.
\]

So \( c_1(\omega_1) = \psi^* c_1(\omega_2) \). Let \( \widetilde{\psi} := \text{pr}_1(\psi(\cdot, z)) \) for some \( z \in S^2 \).

Since \( \psi^* \) preserves \( H^2(X; \mathbb{Z}) \),

\[
\widetilde{\psi}^* c_1(\omega_2) = \psi^* c_1(\omega_2) = c_1(\omega_1).
\]

Since \( \widetilde{\psi} \) is a cohomology (resp. homotopy) equivalence, contradiction.
Counterexamples

Example (Smith, 2000)

Let \( Z := (\mathbb{T}^4_{(x,y,z,t)}, dx dt + dy dz) \# \text{fiber sum} 5(E(1), \omega_0) \) along
\( T_x := \langle x, t \rangle, \ T_y := \langle y, t \rangle, \ T_z := \langle z, t \rangle \) and 2 copies of
\( T_w := \langle x = y = z, t \rangle. \)
Can check that \( T_x, T_w \) are symplectic, while \( T_y, T_z \) are Lagrangian.

Theorem (Smith, 2000)

The simply-connected manifold \( Z \) admits two deformation inequivalent symplectic forms \( \omega_+ \) and \( \omega_- \) obtained by perturbing \( T_z \) “via the opposite orientation”. In fact, \( 3|c_1(\omega_+) \) but not \( c_1(\omega_-) \).
Counterexamples

Theorem (Hirschi-W, ’23)

Let $Z$ be the simply-connected 4-manifold constructed by Smith. Then $\omega_+ \oplus \omega_{std}$ and $\omega_- \oplus \omega_{std}$ on $Z \times S^2$ are deformation inequivalent.

Proof idea: uses the fact that $p_1(Z \times S^2) = p_1(Z) = m\text{PD}([S^2])$ for some $m \neq 0$ and that $p_1$ is preserved by diffeomorphisms up to sign to show that any diffeomorphism of $Z \times S^2$ must lie in $G_{X,S^2}$. 
Example (McMullen-Taubes, 1999)

Let $L := L_1 \sqcup L_2 \sqcup L_3 \sqcup L_4$ consist of four closed geodesics representing the three $S^1$-factors of $\mathbb{T}^3$ and a fourth component satisfying $[L_4] = [L_1] + [L_2] + [L_3]$.

Consider $M := \mathbb{T}^3 \setminus \mathcal{N}(L)$. Let $N$ be the double branched cover of $\mathbb{T}^3$ over $L$ associated to the homomorphism

$$\xi : H_1(M; \mathbb{Z}) \to \{\pm 1\}$$

with $\xi(m_4) = -1$, where $m_4$ is the meridian of $L_4$. 
Then $N$ is fibered, induced from a fibration $\mathbb{T}^3 \to S^1$.

**Definition**

The Euler class of a fibration $\rho : N \to S^1$ only depends on $\alpha = [d\rho] \in H^1(N; \mathbb{Z})$ and is given by

$$e(\alpha) = [s^{-1}(0)] \in H_1(N; \mathbb{Z})/\text{torsion},$$

where $s : N \to \ker(d\rho)$ is a generic section.
Lemma (McMullen-Taubes, 1999)

3D There exist $\alpha_1, \alpha_2 \in H^1(N; \mathbb{Z})$ induced by fibrations $\rho_1, \rho_2 : N \to S^1$ such that $e(\alpha_1)$ and $e(\alpha_2)$ lie in different orbits of $\text{Aut}(\pi_1(N))$-action on $H_1(N; \mathbb{Z})$.

4D Furthermore, let $X := N \times S^1$. One can associate an $S^1$-invariant form $\omega$ to each $\alpha \in H^1(N; \mathbb{Z})$ represented by the differential of a fibration and

$$c_1(X, \omega) = PD_X(e(\alpha) \times [S^1]).$$

Corollary

For any homotopy equivalence $\phi$ of $X$, we have that $\phi_*(e(\alpha_1) \times [S^1]) \neq e(\alpha_2) \times [S^1]$, implying $\phi^*c_1(\omega_2) \neq c_1(\omega_1)$. 
McMullen-Taubes example continued

Remains to show that any diffeomorphism of $X \times S^2$ lies in $\widetilde{G}_{X,S^2}$. This essentially follows from the fact that $X$ is aspherical. Therefore, the McMullen-Taubes construction also gives counterexamples to Donaldson’s 4 – 6 question.
Thank you for listening!