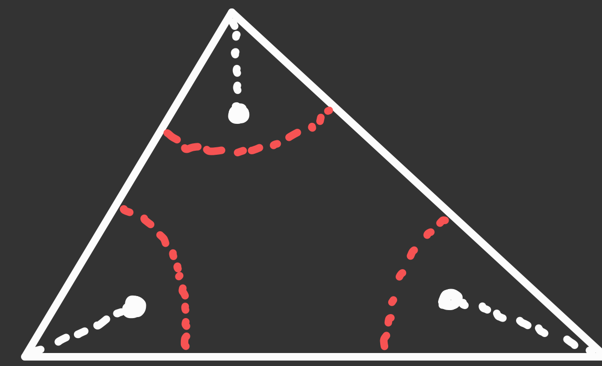
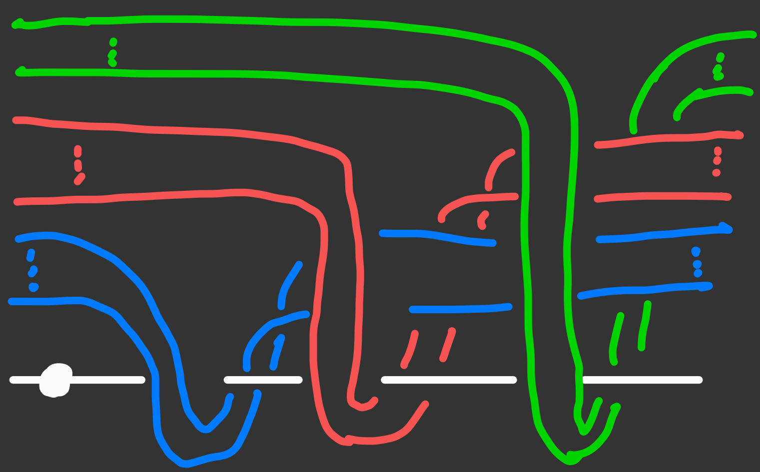
Symplectic Embeddings and small symplectic caps





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Our main task is to build
symplectic manifolds and
symplectic embeddings

but lets start with smooth
embeddings

this is a fundamental problem
in topology:

- isotopy classes of embeddings of S^1 into S^3 is classical knot theory!
- long history of studying when M^m embeds in $\mathbb{R}^n$ or N^n

example: Whitney embedding Th$^{\underline{m}}$

any M^m embeds in $\mathbb{R}^{2n}$

in general this cannot be improved
but some times it can be!

example: Hirsch '61, Wall '65

any 3-manifold embeds in $\mathbb{R}^5$

Corollary: any oriented 3-manifold is the boundary of a 4-manifold!

Cannot do better: not all 3-manifolds embed in $\mathbb{R}^4$ (or any fixed compact 4-manifold Shiomi '91)

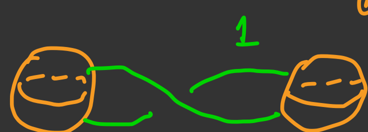
Note: If M embeds in $\mathbb{R}^4$ and M a rational homology sphere, then show M is the boundary of a rational homology ball let's focus on lens spaces

Recall $L(p,q)$ is

not all lens spaces bound homology balls so can't embed in $\mathbb{R}^4$

but some do!

example:

$X =$ 

1-handle $D^1 \times D^3$ glued to D^4 along $(\partial D^1) \times D^3$

2-handle $D^2 \times D^2$ glued along $(\partial D^2) \times D^2$

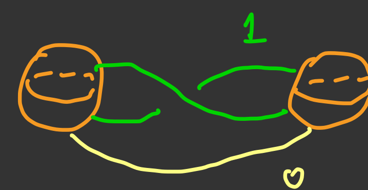
$\partial X = L(4,1)$

but no $L(p,q)$, $|p|>1$, embeds in $\mathbb{R}^4$

Fact: if M^3 embeds in $\mathbb{R}^4$ then $\mathrm{tor}(H_1(M)) \cong G \oplus G$ for some finite abelian group

(Hantzsche '38)

So what is the simplest 4-manifold in which we can embed $L(4,1)$?



$\cup$ 4-handle

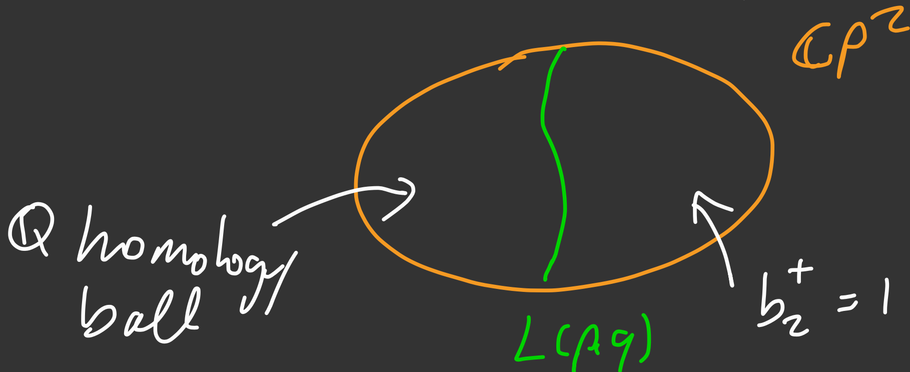
$\Downarrow$ cancel 1,2-handle pair

$\cup$ 4-handle

$=$

$\mathbb{C}P^2$

note: if $L(p,q)$ embeds in $\mathbb{C}P^2$ then $L(p,q)$ or $-L(p,q)$ bound a rational homology ball



Lisca '07: if $L(p,q)$ bounds a $\mathbb{Q}$ homology ball then p is a square (actually gave exact conditions for when $L(p,q)$ bounds $\mathbb{Q}$-homolog ball)

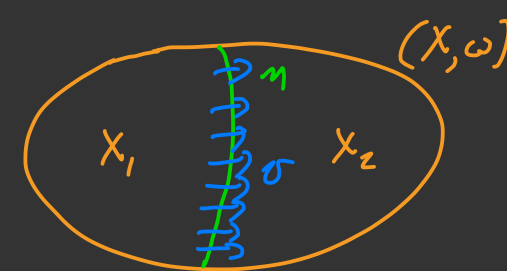
example: $L(3,1), L(5,1)$ don't embed in $\mathbb{C}P^2$

now let's consider the symplectic analog

let (M^3, ξ) be a contact manifold and (X^4, ω) symplectic mfd.

we say (M^3, ξ) embeds in (X^4, ω) as a hypersurface of contact type if $M^3 \subset X^4$ s.t. $\exists$ a vector field v defined near M s.t.

1) $v \pitchfork M$
2) $\mathcal{L}_v \omega = \omega$ — flow of v expands ω
3) $\alpha = \iota_v \omega$ has kernel ξ



Fact: if $M = L(p,q)$ then X_1 must be negative definite

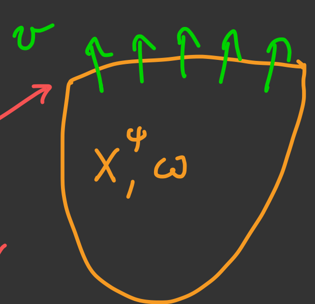
so $X = \mathbb{C}P^2$ then X_1, $\mathbb{Q}$ homolog ball

Question: How can one build symplectic manifolds?

Standard constructions:

1) complex submanifolds of $\mathbb{CP}^n$
2) Symplectic reduction
3) Lefschetz pencils
4) Glue together a symplectic cap and filling of the same contact structure

Recall: a (strong) symplectic filling



called convex boundary

ω closed non-degen

v vector field $\pitchfork \partial X$

$\mathcal{L}_v \omega = \omega$

$\iota_v \omega$ a contact form for ξ on ∂X

a symplectic cap is the same thing but v points into X (boundary concave)

there are lots of symplectic fillings (though not for all contact structures)

E-Honda '02: any contact 3-mfd has a symplectic cap.

unfortunately these caps usually have large topology (eg b_2 large)

so this strategy is no good if one wants to construct small symplectic 4-mfds

example: Is there an symplectic X homeomorpic to, but not diffeomorpic to $\mathbb{CP}^2$? $S^2 \times S^2$?

We would like to

1) build small caps

2) understand symplectic handle attachment

Recall: Eliashberg '90, Weinstein '91

we know how to attach symplectic 0, 1, 2-handles

but ~~cannot~~ attach 3, 4-handles

3) understand the following embedding result from a handlebody perspective

recall a Markov tripple are 3 natural numbers (p_1, p_2, p_3) satisfying

$$p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3$$

eg

$$(1,1,1) \to (1,1,2) \to (1,2,5) \to (2,5,29)$$
$$(1,2,5) \to (1,5,13)$$

(p_1, p_2, p_3) tripple then so is $(p_1, p_2, 3p_1p_3 - p_2)$

given (p,q) let $B_{p,q}$ be the symplectic $\mathbb{Q}$-homology ball

p strands

$\partial B_{p,q} = L(p^2, pq-1)$ (lens space)

Viana '17: given a Markov tripple (p_1, p_2, p_3)

$\exists\, q_i$ such that $B_{p_1,q_1}, B_{p_2,q_2}, B_{p_3,q_3}$ embed disjointly in $\mathbb{C}P^2$.

(Evans-Smith '18: showed these are only contact embeddings of lens spaces in $\mathbb{C}P^2$)

Viana used almost toric geometry and "nodal slides" and "transfer the cut" to construct the embeddings

Remark:

1) Lisca '08 showed that the only universally tight contact structures on $L(p,q)$ bounding $\mathbb{Q}$-homology balls are the ones above

E-Roy '21, Christian-Li showed that no virtually overtwisted contact structure bounds a $\mathbb{Q}$ homology ball

2) lots of lens spaces don't embed as contact type hypersurfaces in $\mathbb{CP}^2$, eg. $L(9,5) = \partial B_{3,2}$

3) smoothly more lens spaces embed

Owens '19:

$\boxed{\text{$B_{F(2n+1),F(2n-1)}$ embeds in $\mathbb{CP}^2$ but not symplectically for $n>1$}}$

example $L(25,14)$ embeds but not as a hypersurface of contact type

Lisca-Parma '23: gave lots more examples

Question: exactly which lens spaces embed in $\mathbb{CP}^2$?

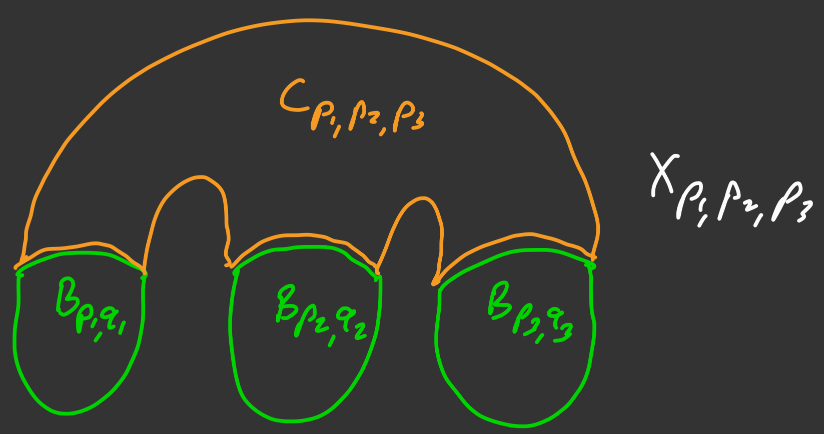
Th^{m} (E-Min-Piccirillo-Roy)

given a Markov tripple (p_1, p_2, p_3) and (q_1, q_2, q_3) as in Viana's result we can give an explicit handlebody construction of a symplectic cap C_{p_1,p_2,p_3} for

$$L(p_1^2, p_1q_1-1) \amalg L(p_2^2, p_2q_2-1) \amalg L(p_3^2, p_3q_3-1)$$

and $b_2(C_{p_1 p_2 p_3}) = 1$

We can glue in the B_{p_i, q_i} above to get a close symplectic manifold X_{p_1,p_2,p_3}



note: one can check $\pi_1(X_{p_1p_2p_3}) = 1$ and $H_2(X_{p_1p_2p_3}) \cong \mathbb{Z}$ so Freedman says $X_{p_1p_2p_3}$ is homeomorphic to $\mathbb{C}P^2$.

Is it exotic!

Th^{m} (E-M-P-R):

$X_{p_1p_2p_3}$ is diffeomorphic to $\mathbb{C}P^2$

Lisca-Parma '23 proved the B_{p_i, q_i} above smoothly embed in $\mathbb{C}P^2$ using an explicit handle body construction (Horizontal decompositions) along the way we will also be able to give an <u>explicit handlebody picture</u> for the almost toric pictures and constructions of Viana

Main Ingredients:

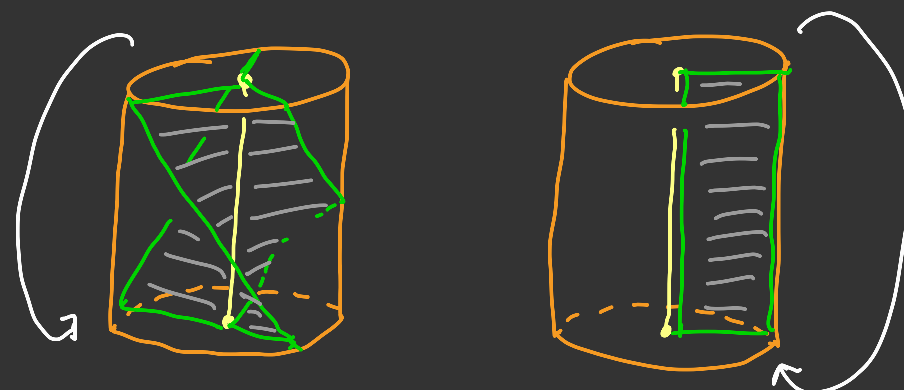
1) a construction of Gay
   (+ slight generalization)
2) an understanding of
   non-loose torus knots
   (E-Min-Mukherjee)
3) lots of handle calculus!

let's get started!

Open book decompositions and Gay's handle

a rational open book for a 3-manifold M is a pair (B, π) where B is a link in M and $\pi : (M - B) \to S^1$ is a fibration such that each component of $\partial \overline{\pi^{-1}(\theta)}$ wraps around a component of B (if $\pi^{-1}(\theta)$ is an embedded surface with boundary B then can drop rational)

near component of B see:



example: let H = Hopf link in S^3

$S^3 - H = T^2 \times (0,1)$, fiber T^2 by any slope curve C but 0 or ∞

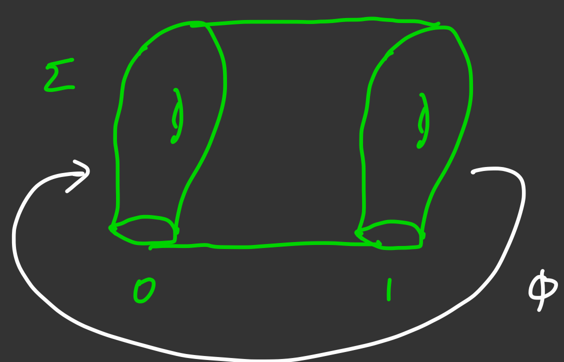
then $C \times (0,1)$ fibers $S^3 - H$

a contact structure ξ on M is supported by (B, π)
if 1) B is positively transverse to ξ
2) $\exists$ a contact form α for ξ s.t. $d\alpha > 0$ on $\pi^{-1}(\theta)$

Thurston-Winkelnkemper 75, Baker-E-Van Horn-Morris '12 (rational case)

show every openbook supports a (unique) contact structure

note: since $(M - B)$ fibers over S^1 it is a mapping torus of a diffeomorphism $\phi : \Sigma \to \Sigma$ where $\Sigma = \overline{\pi^{-1}(\theta)}$



$\cong M - B$

so ϕ determines a 3-manifold and a contact structure, ϕ is called the monodromy

Th$^{\underline{m}}$ (Gay '02, E-M-P-R for $\mathbb{Q}$ case)

let (B, π) be an open book for (M, ξ)

$B = \{B_1, \ldots, B_k\}$

$\{n_1, \ldots, n_k\}$ integers larger than page slope on B_i

$W = ([0,1] \times M)$ with 2-handles attached to $\{1\} \times M$ along B_i with framing n_i

Then W has a symplectic structure with both boundary components concave

W is a cobordism from (M, ξ) to the result of $-n_i$ surgery on $\overline{B}_i$ in $(-M, \xi_{\overline{B}})$

contact str. supported by $\overline{B}$ in $-M$

so if (M, ξ) is fillable then we can use W to build a symplectic cap!

e.g. $(S^3, \xi_{std}) = \partial(B^4, \omega_{std})$ so th$^{\underline{m}}$ gives b_2 small caps

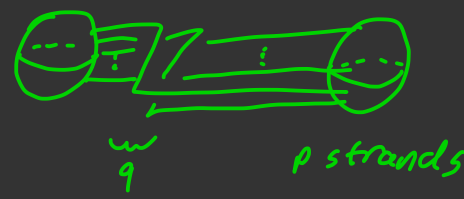
note: the monodromy of $(-M, \xi_{\overline{B}})$ is ϕ^{-1}

so if ξ is tight $\xi_{\overline{B}}$ will **not** be tight

e.g. positive torus knots in S^3 support the unique tight structure but their mirrors support overtwisted contact strs.

surgeries on overtwisted contact str are usually overtwisted, so how to use this!

recall: If $B_{p,q}$ is

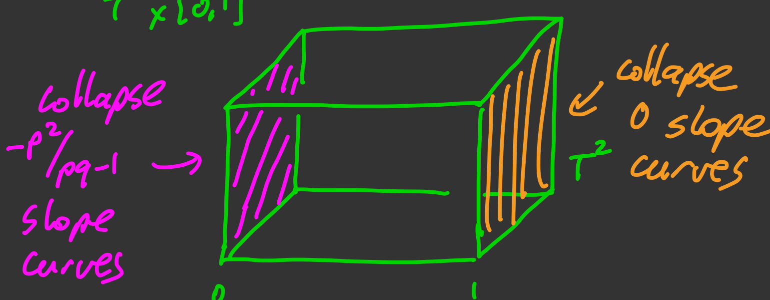


then $\partial B_{p,q} = L(p^2; pq-1)$

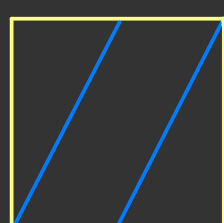
with its universally tight contact structure

What is $L(p^2; pq-1)$:

$T^2 \times [0,1]$



let γ be a $r/s > 0$ curve on $T^2 \times \{1/2\}$



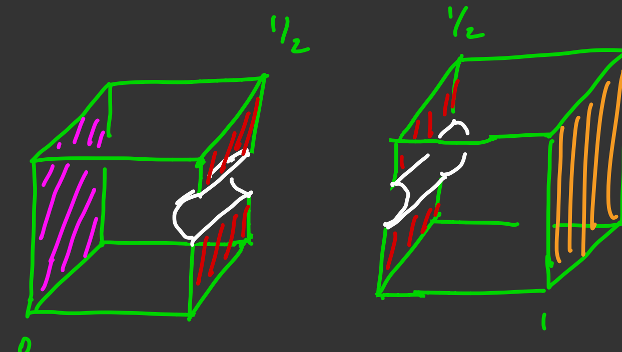
$T^2 \times \{1/2\}$

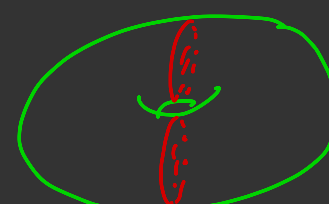
add a 2-handle to $[0,1] \times L(p^2; pq-1)$ along γ in $\{1\} \times L(p^2; pq-1)$ with framing $\mathcal{F}$ = framing from $\{1/2\} \times T^2$ to get W

$\partial W = L(p^2; pq-1) \cup M$

$M = \mathcal{F}$ framed Dehn surgery on γ

in M we see $(\{1/2\} \times T^2) - (\text{nbhd } \gamma)$ union 2 meridional disks





surgery torus

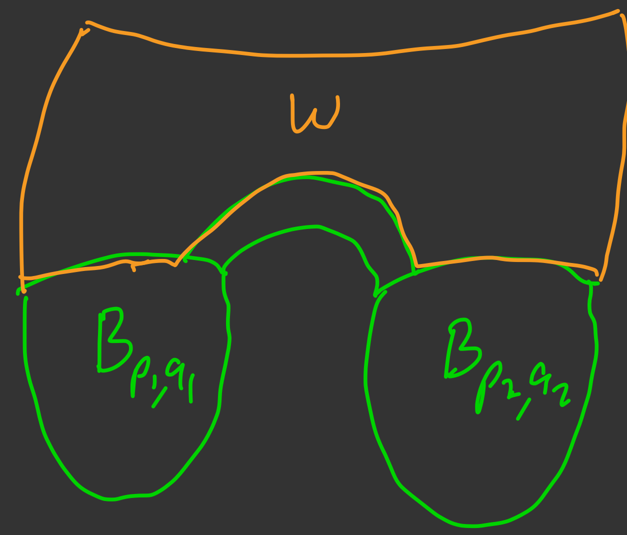
this is a sphere S^2!

so $M = M' \# M''$

not hard to see one of these is $-L(r,s)$ and other is $L(m,n)$ for some m,n

can order the p_i's so $(p,q) = (p_3^2, p_3 q_3 - 1)$
$(r,s) = (p_1^2, p_1 q_1 - 1)$ and $(m,n) = (p_2^2, p_2 q_2 - 1)$

smoothly we can add a 1-handle to $B_{p_1,q_1} \cup B_{p_2,q_2}$ and then glue W to it



now remove $B_{p_1,q_1} \cup B_{p_2,q_2}$ to get $C_{p_1 p_2 p_3}$

can we do this symplectically?

Yes for green part:

B_{p_i,q_i} symplectic

1-handles can be attached symplectically for W:

- $[0,1] \times L(p_3^2, p_3 q_3 - 1)$ is part of the symplectization of standard tight str ξ_{ut}
- the knot γ is a $(p_1^2, p_1 q_1 - 1)$-torus knot in $L(p_3^2, p_3 q_3 - 1)$
  easy to check γ supports ξ_{ut}

so Gay + E theorem says we can attach handle symplectically!

$\therefore$ W is symplectic!

But: does the "upper" boundary of W have the right contact structure?

contact structure is obtained from surgery on $\xi_{\bar{\gamma}}$

unfortunately $\xi_{\bar{\gamma}}$ is overtwisted...

so we need to understand non loose knots in lens spaces

Nonloose torus knots in lens spaces

E-Min-Mukherjee '22 classified all Legendrian and transverse knots in overtwisted contact structures on S^3

the part we need on lens spaces generalizes this

let's consider the S^3 case for now

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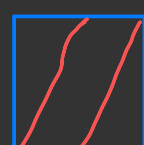
complement tight

understand non-loose (p,q) torus knots w/ $tb = pq$

curves on torus $S^1 \times S^1$

$\updownarrow$

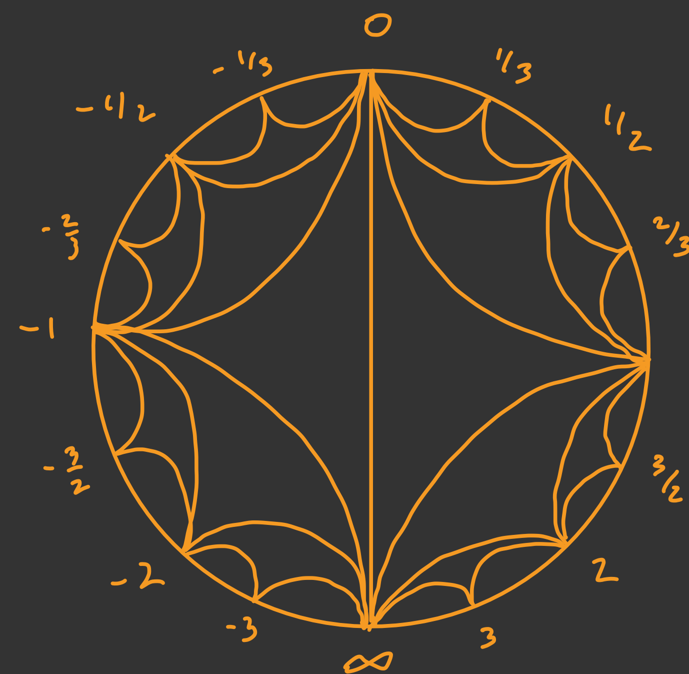
$\mathbb{Q} \cup \{\infty\}$



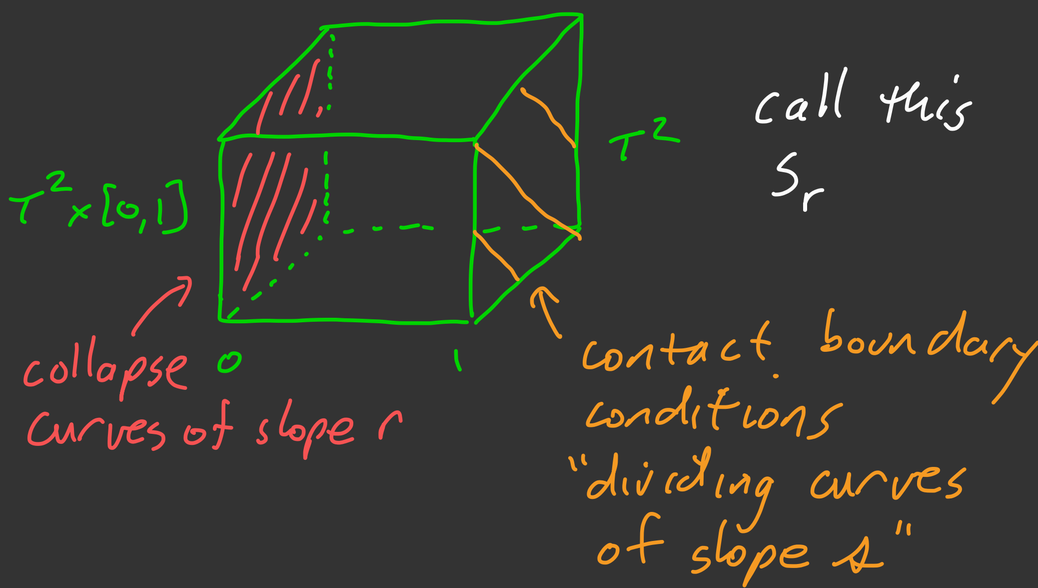
$\updownarrow$

2

keep track of curves with Farey Graph



think of solid torus as



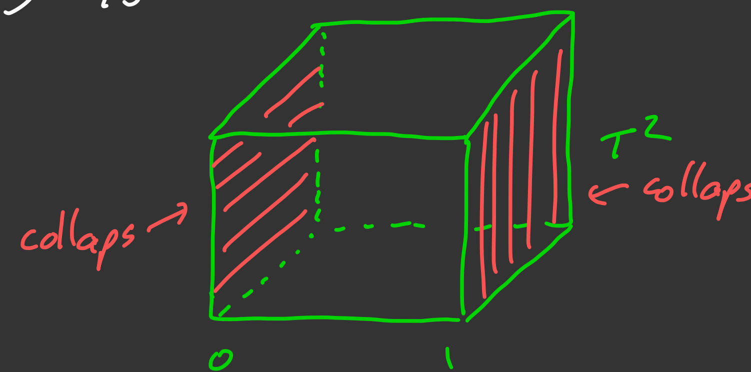
call this S_r

Th^m (Giroux '00, Honda '00)

tight contact structures on S_r with dividing slope s are given by taking a minimal path in the Farey graph from r clockwise to s and decorating the edges with $\pm$ (except 1^{st})

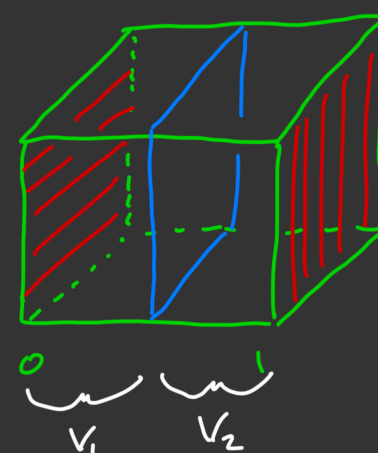
(can do same with $T^2 \times \{1\}$ collapsed)

S^3 is

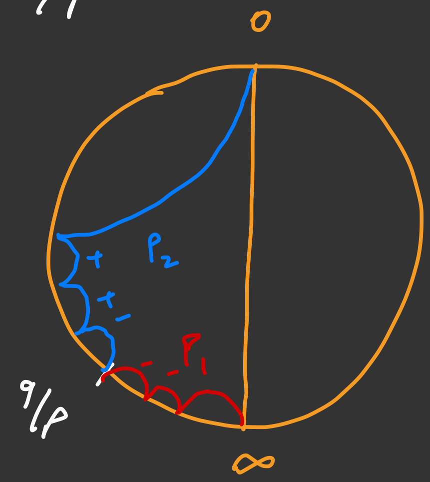


can take a $T^2 \subset S^3$

splits S^3 into 2 solid tori V_1, V_2



put dividing curves on T^2 of slope q/p



the paths P_1, P_2 give contact strs on V_1 and V_2 $\therefore$ a contact str on S^3

Th$\underline{\text{m}}$:

$\boxed{\text{"any" non-loose } (p,q)\text{-torus knot (with no torsion) sits on a } T^2 \text{ for some choice of } P_1, P_2}$

Paths determined by continued fractions ...

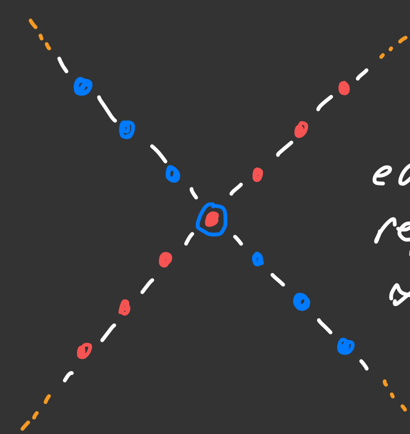
non-loose (p,q)-torus knots were studied by Geiges-Onaran '20

Matković '20 for some (p,q) and some values of Thurston-Bennequin invariant

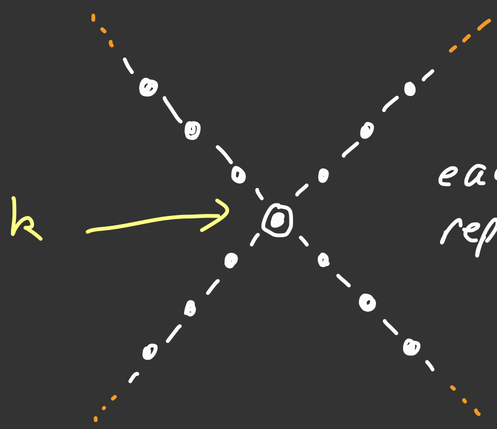
E-M-M gave a complete classification

if we plot possible rotation numbers and tb invariants then can see

for $pq>0$:

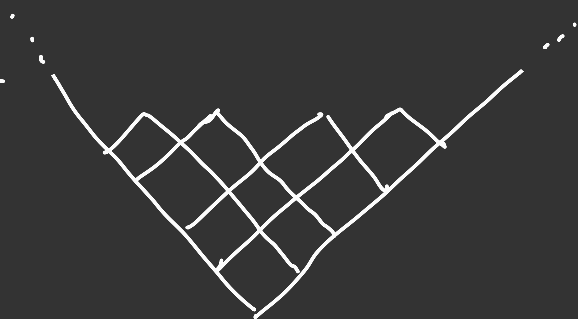


each dot represents ∞ly many Legendrians

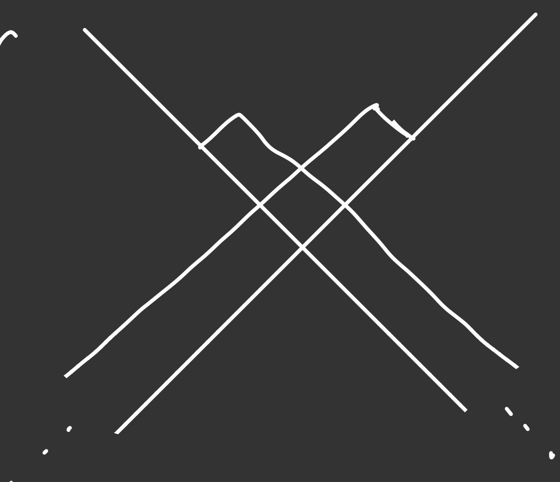


each dot represents one Legendrian

or



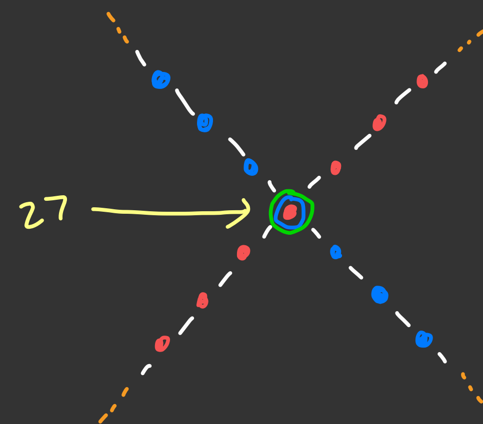
or



for $pq<0$:

see first 2 "X s"

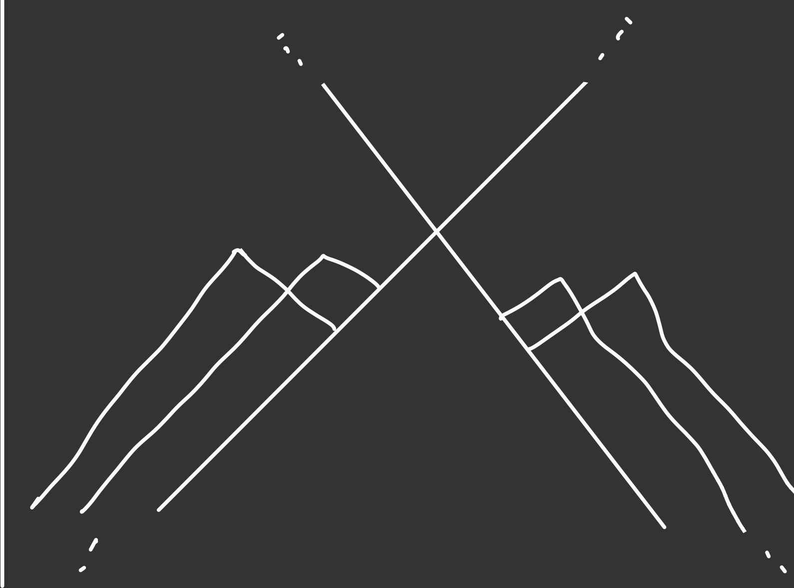
or



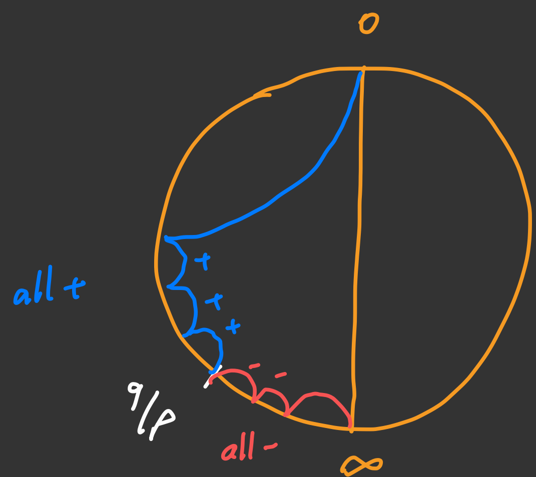
each •, • dot represents one Legendrian

○ is a unique Legendrian L s.t.
$S_+(L) =$ •
$S_-(L) =$ •

or



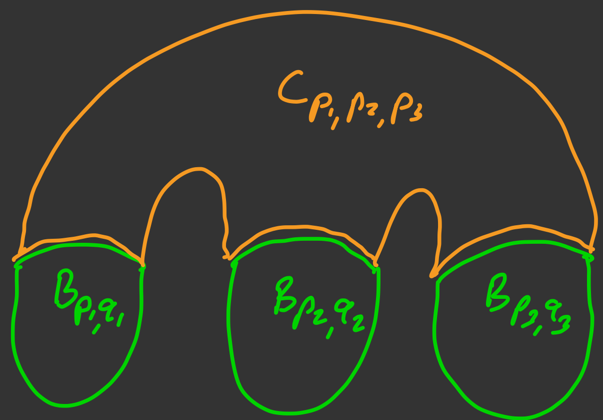
E-M-M showed a negative (p,q)-torus knot supports the contact structure



and pq surgery on (p,q)-knot gives # of universally tight contact structures!

so we can glue our symplectic pieces together to get $C_{p_1 p_2 p_3}$ as claimed in the main theorem!

But is X_{p_1,p_2,p_3}

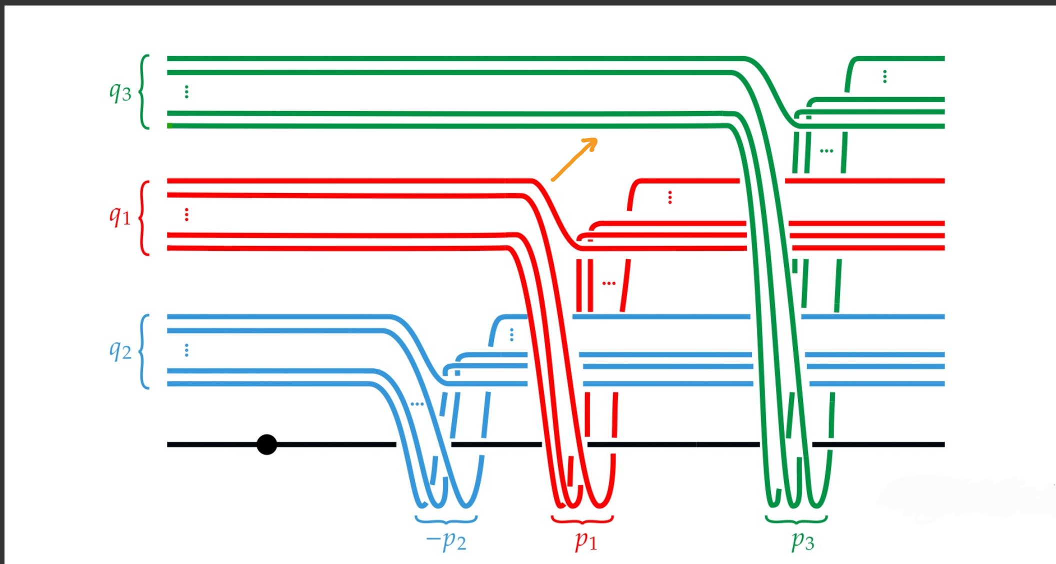


really $\mathbb{CP}^2$?

There are 2 ways to show this

1. Quote thm from Lisca-Parma you can write $X_{p_1 p_2 p_3}$ as a "horizontal decomposition"
2. Find a handlebody description of almost toric pictures an inductively prove.

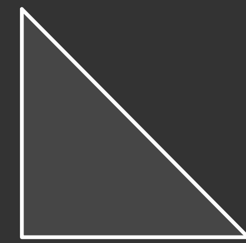
here is a picture of $X_{p_1 p_2 p_3}$:

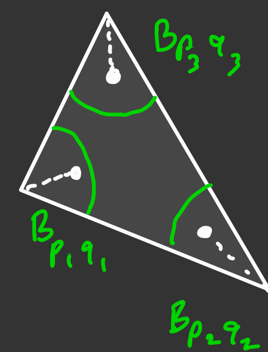


union 3-handle
and 4-handle

this handle slide is inspired by
almost tori geometry

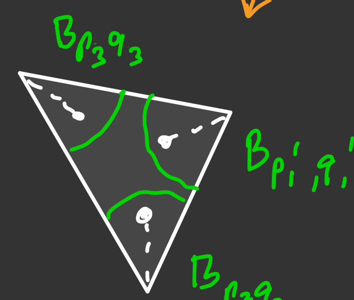
toric picture of $\mathbb{C}P^2$





almost toric
picture of $\mathbb{C}P^2$

"transfer the cut"



$p_1' = 3p_1p_3 - p_2$

We show how to do this using
handle calculus

Base case $X_{1,1,1}$ is

What's next?

① easy to construct embeddings of other lens spaces into $\mathbb{C}P^2 \#_n \overline{\mathbb{C}P}^2$ using same approach.

What would an optimal result be?

What about embeddings into $S^2 \times S^2$?

② Build other small caps

In particular, can one find small exotic symplectic 4-manifolds?

Thanks

for your

Attention