Equivariant Floer homotopy via Morse-Bott theory

joint work with Laurent Côté

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December 22, 2023

Overview

1. The homotopy type from Morse theory
2. Morse-Bott flow categories and their geometric realization
3. Equivariant Floer homotopy type
4. Applications

Homotopy type from Morse theory

Let $f : N \to \mathbb{R}$ be a Morse function, and g be a metric. Given $x, y \in crit(f)$, let $\mathcal{M}_f^\circ(x, y)$ denote the moduli of trajectories of $-grad_g(f)$ from x to y (moduli translation), and $\mathcal{M}_f(x, y)$ denote its compactification by broken gradient trajectories. For generic g, $\mathcal{M}_f(x, y)$ is a manifold with corners.

- (classical) one can calculate $H_*(N)$ from the data of 0 and 1-dimensional moduli $\mathcal{M}_f(x, y)$
- (Cohen–Jones–Segal) one can actually recover the entire homotopy type from the knowledge of all moduli spaces $\mathcal{M}_f(x, y)$ and (stable) trivialization of their tangent bundles

This is the basis of Floer homotopy theory.

Homotopy type from Morse theory (cont'd)

How to reconstruct homotopy type of N from the data of critical points and moduli?

Example

Assume f has two critical points $\{x, y\}$ such that $\mathcal{M}_f(x, y) \neq \emptyset$. For example, $f : S^n \to \mathbb{R}$ is the height.
Let $V_x \subset T_x N$, resp. $V_y \subset T_y N$ be a maximal negative definite subspace for the Hessian, i.e. "descending directions". Then, $\underline{V_x} \simeq T_{\mathcal{M}_f(x,y)} \oplus \underline{\mathbb{R}} \oplus \underline{V_y}$. This identity, and Pontryagin-Thom collapse for $\mathcal{M}_f(x, y)$ allows one to recover the stable homotopy type of N.

Remark

The construction related to Floer homotopy theory require the data of stable trivializations, but the earliest result of Cohen-Jones-Segal recovers the homeomorphism type without this data.

Homotopy type from Morse theory (cont'd)

This is the basis of Floer homotopy theory.
Cohen–Jones–Segal define a category $\mathcal{M}_f$ such that

- $ob(\mathcal{M}_f) = crit(f)$
- morphisms $\mathcal{M}_f(x, y)$ is as defined above
- composition is concatenation

The key property is that the composition $\mathcal{M}_f(x, y) \times \mathcal{M}_f(y, z) \hookrightarrow \partial\mathcal{M}_f(x, z)$ is injective and the images of composition maps cover $\partial\mathcal{M}_f(x, z)$. This is example of a *flow category*.
For a given flow category $\mathcal{M}$, fixing (stable) framings for each $\mathcal{M}(x, y)$ leads to a (stable) homotopy type $|\mathcal{M}|$, and

Theorem (Cohen-Jones-Segal)

$|\mathcal{M}_f|$ is equivalent to N.

Equivariant case?

Morse theory is not compatible with group actions. If G is a compact Lie group acting on N, typically there are no equivariant Morse functions (and Morse-Smale pairs).

Example

If $G \neq 1$ is connected, it fixes the critical points of an equivariant Morse function. On the other hand, if the action is also free, this is not possible.

As a result, we need to go beyond Morse functions, namely to Morse–Bott functions.

Morse–Bott functions

Recall: A *Morse–Bott function* is a function with local expression $f(x_1, \dots, x_n) = c \pm x_1^2 \pm \dots x_k^2$. Its critical set is a disjoint union of manifolds, which we will refer as the *critical manifolds*.

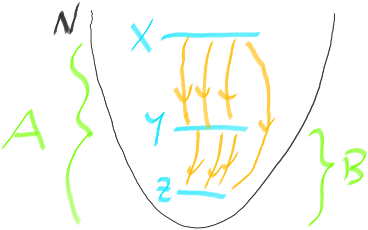
Example

Let G act freely on N, and let $f_0 : N/G \to \mathbb{R}$ be a Morse function. Then, the composition $N \to N/G \xrightarrow{f_0} \mathbb{R}$ is Morse–Bott. The critical manifolds are given by the pre-images of the elements of $crit(f_0)$ and they are G-torsors.

Fix a metric. For two critical manifolds X, Y, let $\mathcal{M}_f(X, Y)$ denote the moduli of broken trajectories from a point on X to a point on Y. There are natural evaluation maps $X \leftarrow \mathcal{M}_f(X, Y) \to Y$.
Banyaga: if the metric is Morse-Bott-Smale, then one can calculate $H_*(N)$ from the data of critical manifolds and moduli of gradient trajectories (+some orientations).

Recovering the homotopy type from Morse-Bott theory

Let $f : N \to \mathbb{R}$ be a Morse-Bott function, $X, Y, Z, \ldots$ critical manifolds, A, B sublevel sets.



Let $T^d X$ denote the descending bundle on X (i.e. the descending directions). Observe, $A/B \simeq X^{T^d X}$, the Thom space of $T^d X$.

Recovering the homotopy type from Morse-Bott theory (cont'd)

Hence, we have a Puppe sequence

$$B_+ \hookrightarrow A_+ \twoheadrightarrow X^{T^d X} \to \Sigma B_+$$

and A_+ is equivalent to the (homotopy) fiber of $X^{T^d X} \to \Sigma B_+$. By iterating this process, we find out that

$$A_+ \simeq hofib(X^{T^d X} \to hofib(\Sigma Y^{T^d Y} \to hofib(\Sigma^2 Z^{T^d Z} \to \dots)))$$

i.e. we can build N from the Thom spaces $X^{T^d X}$, $Y^{T^d Y}$, ...

Relative framings and correspondence maps

- Banyaga uses the correspondence $X \leftarrow \mathcal{M}_f(X, Y) \rightarrow Y$ to produce maps $C_*(X) \rightarrow C_*(Y)$, and hence a double complex computing the homology
- this requires choice of orientations, or otherwise one would need homology with local coefficients
- the space level analogue of an orientation on a bundle is stable trivialization
- the space level analogue of homology with local coefficients is a Thom space X^ξ

Relative framings and correspondence maps

Let $V_X := T^d X$. It is easy to show that

$$V_X \oplus T_X \simeq T_{\mathcal{M}_f(X,Y)} \oplus \underline{\mathbb{R}} \oplus V_Y$$

This is a framing on $T_{\mathcal{M}_f(X,Y)}$ relative to X and Y. It allows one to write correspondence maps

$$X^{V_X} \to \mathcal{M}_f(X,Y)^{V_X + T_X - T_{\mathcal{M}_f(X,Y)}} = \mathcal{M}_f(X,Y)^{V_Y + \mathbb{R}} \to Y^{V_Y + \mathbb{R}}$$

Framed Morse–Bott flow categories

We organize this data in a structure called a *framed Morse–Bott flow category*. Namely, let $\mathcal{M}_f$ denote the category such that

- $ob(\mathcal{M}_f) = crit(f)$
- given components $X, Y \subset crit(f)$ the morphisms from a point of X to a point of Y is the same as $\mathcal{M}_f(X, Y)$

Observe that the composition is defined on $\mathcal{M}_f(X, Y) \times_Y \mathcal{M}_f(Y, Z) \to \mathcal{M}_f(X, Z)$. These are injective and their images cover the boundary of $\mathcal{M}_f(X, Z)$.

Framed Morse–Bott flow categories

Definition (Zhou, Côté-K.)

A *framed Morse–Bott flow category* $\mathcal{M}$ is a (non-unital, directed) topological category such that

- $ob(\mathcal{M})$ is a disjoint union of smooth manifolds
- $\mathcal{M}(X, Y)$ is a smooth manifold with corners
- additional properties

Each component $X \subset ob(\mathcal{M})$ is equipped with a (virtual) bundle V_X and as part of the data we have isomorphisms

$$V_X \oplus T_X \simeq T_{\mathcal{M}(X,Y)} \oplus \underline{\mathbb{R}} \oplus V_Y$$

Geometric realization

By the framing identity, $V_X \oplus T_X \simeq T_{\mathcal{M}(X,Y)} \oplus \underline{\mathbb{R}} \oplus V_Y$ we have maps $X^{V_X} \to \Sigma Y^{V_Y}$ as observed.
Hence, we obtain a "chain complex" in spaces (spectra) in $X^{V_X}, Y^{V_Y}, \dots$

Definition (Côté-K.)

Let $|\mathcal{M}|$ denote the realization of this "chain complex". We call $|\mathcal{M}|$ the *geometric realization* of the framed flow category $\mathcal{M}$.

Example

When there are two components X, Y, the realization is the cone/homotopy fiber of $X^{V_X} \to \Sigma Y^{V_Y}$ (up to a shift). When there are more, it is an iterated cone.

Main result in differential topology

Theorem (Côté-K.)

$|\mathcal{M}_f|$ is stable homotopy equivalent to N. Moreover, if there is action of a compact group G on N, and (f, g) are equivariant, this is a G-equivariant equivalence.

When the group action is free, one can always lift a Morse-Smale pair on N/G. When the action is not free, one can replace N by $N \times EG$.

Remark

The realization depends on the framing. For instance, on $\mathcal{M}_f$, one replaces the framing above by $V_X = T^d X \oplus \nu|_X$ (where ν is a (virtual) bundle on N), one obtains the Thom space (Thom spectrum, really) N^ν.

Equivariant Hamiltonian Floer homology

Let M be a Liouville manifold. Assume T_M is symplectically stably trivial. Consider $S^\infty = \{(z_1, z_2, \dots) : \sum |z_i|^2 = 1\}$, and the equivariant Morse–Bott function $h : S^\infty \to \mathbb{R}$ given by $h(z_1, z_2, \dots) = \sum k|z_k|^2$. Fix a slope a, and choose (H, J) such that

- $H : S^\infty \times S^1 \times M \to \mathbb{R}$ is a family of Hamiltonians of slope a and J is a family of cylindrical almost complex structures parametrized by $S^\infty \times S^1$
- (H, J) is S^1-equivariant with respect to the diagonal action on $S^\infty \times S^1$
- given $s \in crit(h)$, H_s is non-degenerate
- (H, J) satisfies regularity assumption

Equivariant Hamiltonian Floer homology (cont'd)

Consider the category $\mathcal{M}_{h,H}$ such that

- $ob(\mathcal{M}_{h,H}) = \{(a,x) : a \in crit(h), x \in orb(H_a)\}$
- $\mathcal{M}_{h,H}((a,x),(b,y)) = \{(\gamma,u)\}$, where γ is a negative gradient trajectory of h from a to b and u is a holomorphic cylinder from x to y "over γ"

Theorem (Large)

$\mathcal{M}_{h,H}$ is an equivariant Morse–Bott flow category.

Strictly speaking, this is an extension of Large's result over S^∞.
One can also equivariantly extend stable framings of Large, making $\mathcal{M}_{h,H}$ framed.

Definition

Let $HF_S(M,\mathbb{S})$ denote the S^1-equivariant homotopy type $|\mathcal{M}_{h,H}|$, and let $SH_S(M,\mathbb{S})$ denote its colimit as slope goes to infinity.

Restriction maps and Viterbo isomorphism

Let $Q \subset M$ be an exact, compact Lagrangian. Then we have

Theorem (Côté-K.)

There are S^1-equivariant restriction maps $SH_{\mathbb{S}}(M, \mathbb{S}) \to (\mathcal{L}Q)^{-W_{mas}}$, where W_{mas} is a natural virtual bundle on $\mathcal{L}Q$, and this is an equivalence of (Borel) equivariant spectra when $M = T^*Q$.

To have an equivalence of Borel equivariant spectra, one only needs an equivariant map that is also an equivalence (this is not the case for other notions of equivariant spectra, such as genuine equivariant spectra).

Restriction maps and Viterbo isomorphism (cont'd)

Once one has the equivariant map, checking Viterbo isomorphism boils down to classical Viterbo isomorphism and the Hurewicz theorem. Namely, the map $SH_S(T^*Q, \mathbb{S}) \to (\mathcal{L}Q)^{-W_{mas}}$ induces an isomorphism on homology; hence, its cone has vanishing homology. The cone also splits into direct sum of bounded below spectra; hence, by Hurewicz, it is contractible.

Remark

The twist by $-W_{mas}$ is expected by string topology as well: $SH_S(T^*Q, \mathbb{S})$ carries an associative product, which should be compatible with the string product. However, the latter is defined only on $\mathcal{L}Q^{-TQ}$ (Cohen-Jones).

Maps from flow categories to spaces: P-valued modules

Let $\mathcal{M}$ be a framed (for simplicity standard, non-Morse–Bott) flow category, P be a space, ν be a (virtual) bundle on P.

Definition

A *framed P-valued module over $\mathcal{M}$* is an assignment $x \to \mathcal{N}(x)$ of smooth manifold with corners to each $x \in ob(\mathcal{M})$, and composition maps $\mathcal{M}(x,y) \times \mathcal{N}(y) \hookrightarrow \partial\mathcal{N}(x)$ satisfying properties. It also carries

- evaluation maps $\mathcal{N}(x) \to P$ compatible with the composition
- relative framing $V_x \simeq T_{\mathcal{N}(x)} \oplus \nu$

This notion has extensions into Morse–Bott and equivariant case.

Proposition

A framed P-valued module gives rise to a map $|\mathcal{M}| \to P^{\nu}$.

Maps from flow categories to spaces: P-valued modules (cont'd)

Example

Let $f : N \to \mathbb{R}$ be Morse, and define $\mathcal{N}(x)$ to be the (compactified) descending manifold, or the compactified moduli of half gradient trajectories asymptotic to x. Here, $P = N$, and the evaluation map is the end point. We obtain the map $|\mathcal{M}_f| \to N$ (to $\Sigma^\infty N_+$, really).

Example

Moduli of holomorphic half cylinders with boundary on the Lagrangian Q gives a $\mathcal{L}Q$-valued module. The evaluation map is the boundary on Q. To frame, one needs to take $\nu = -W_{mas}$. This gives rise to the map to $(\mathcal{L}Q)^{-W_{mas}}$.

Applications I (joint work in progress with Côté):

The symplectic cohomology is not sensitive to the homotopy type of M. However, Albers-Cieliebak-Frauenfelder, Zhao showed how to recover rational homology from the filtered equivariant homotopy type of the symplectic cohomology.

$SH(M, \mathbb{S})$ also has a natural filtration. We prove

- one can recover the complex K-theory of M from the filtered, equivariant homotopy type of $SH(M, \mathbb{S})$
- one can also recover Morava K-theories, as well as integral homology of M

These are both as ungraded vector spaces.

Applications I (joint work in progress with Côté, cont'd):

Given complex oriented cohomology theory R, we define *the completed Tate cohomology* of the filtered equivariant spectrum $SH_S(M, \mathbb{S})$ as follows:

1. take the homotopy quotient of $F^p SH_S(M, \mathbb{S})$ by S^1 and apply cohomology R^*
2. this is a module over $R^*[[u]] = R^*(\mathbb{CP}^\infty)$, invert u
3. invert the elements $[n]_R(u) \in R^*[[u]]$ for all n (this is the n^{th} power of u with respect to a formal group law)
4. take inverse limit, as $p \to \infty$

We denote this invariant by $\widehat{R}^*_{S^1}(SH_S(M, \mathbb{S}))$ and call it the *completed Tate cohomology* (somehow a misnomer).

Applications I (joint work in progress with Côté, cont'd):

Note that it depends on the filtration in a very loose sense. The effect of inverting $[n]_R(u) \in R^*[[u]]$ is to kill the non-constant orbits as

$$R^*_{S^1}(S^1/C_n) = R^*(BC_n) = R^*[[u]]/[n]_R(u)$$

Hence, the invariant depends only on the topology of M. For example,

- if R is cohomology with coefficients in $\mathbb{Q}$ or $\mathbb{Z}$, it is $H^*(M, \mathbb{Q}((u)))$
- if R is cohomology with coefficients in $\mathbb{F}_p$, it is 0
- if R is complex K-theory, the completed Tate cohomology and $KU^*(M)((u))$ agree after completion at every prime
- if R is a Morava K-theory, the completed Tate cohomology is $R^*(M)((u))$

The last item can be used co recover information about torsion cohomology. Speculatively, finer information (such as stable homotopy groups) can be recovered using chromatic homotopy theory.

Applications II (joint work in progress with Côté):

Expectation: $SH(M, \mathbb{S})$ is the *topological Hochschild homology* of the (spectral, wrapped) Fukaya category of M. Hence, it carries a *cyclotomic structure*, i.e. an S^1-action, and equivariant Frobenius maps.
We have a proposal for the Frobenius map and checking S^1-equivariance is very easy thanks to the description above.
This is part of a long term project to understand THH (and K-theory of Fukaya categories).

Application III (joint work in progress with Côté and Mak):

Expectation from physics: given Hamiltonian G-action on M, we expect an algebra map $R_*^G(\Omega G) \to SH_G^*(M, R)$, where $R_*^G(\Omega G)$ is the pure Coulomb branch algebra, and R is homology, complex K-theory or elliptic cohomology.
We can construct a G-equivariant model for $SH(M, \mathbb{S})$ using the framework above, and its equivariant R-homology is $SH_G^*(M, R)$. We work on defining the algebra map using parametrized homotopy theory.

Thank you!