Hofer Norms on Braid Groups

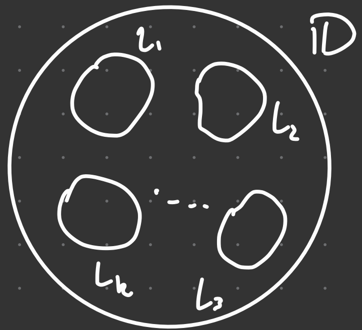
&

Quantitative Heegaard-Floer Homology

Setup

$\underline{L}$ collection of k disjoint, non-nested, embedded circles in $\mathbb{D}$.

$$\text{Area}(\mathbb{D}) = 1.$$



$$\text{Ham}_{\underline{L}}(\mathbb{D}) = \{ \varphi \in \text{Ham}_c(\mathbb{D}) \mid \varphi(\underline{L}) = \underline{L} \}$$

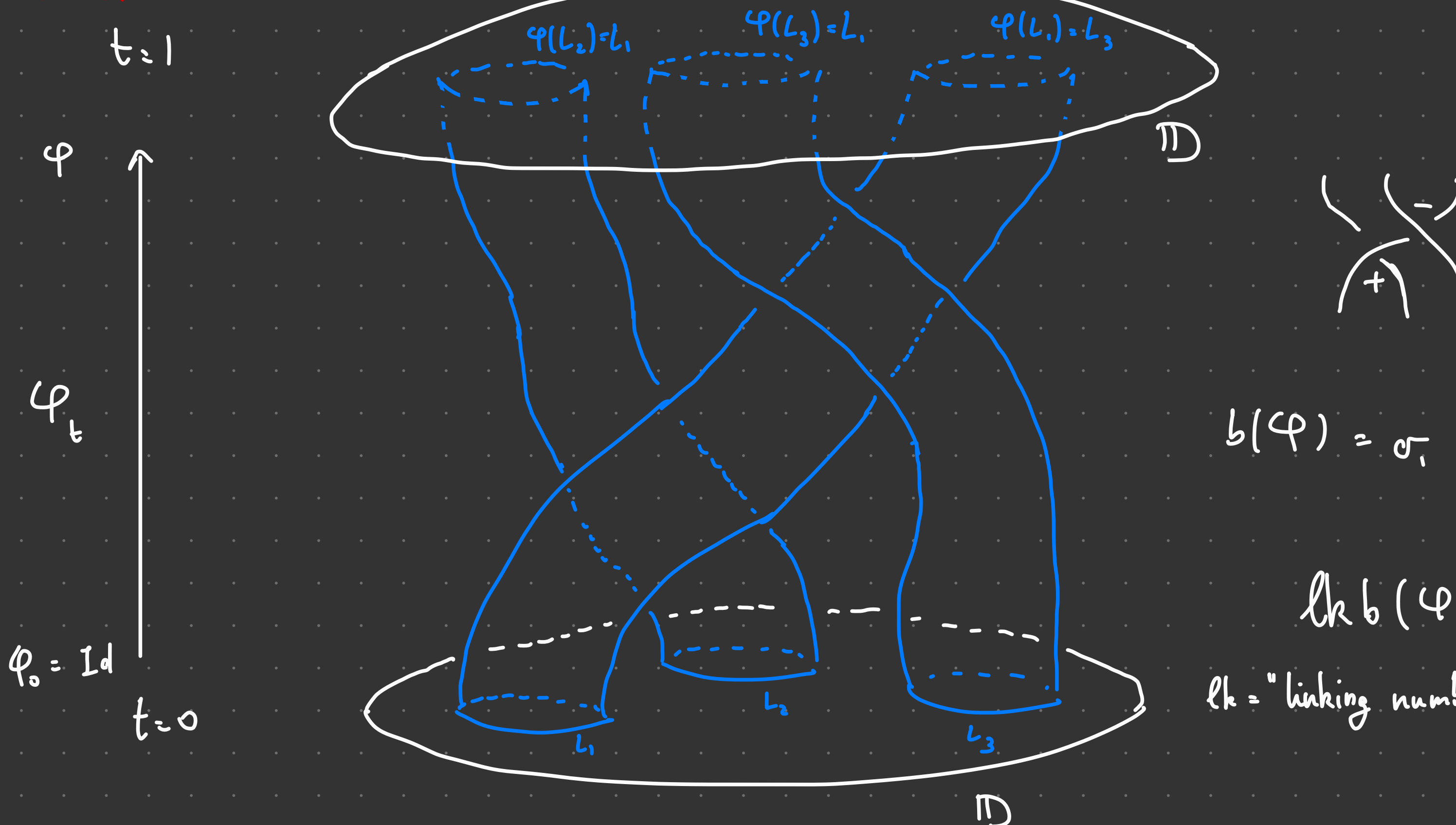
$\rightsquigarrow$ braid type function (surjective morphism of groups if circles bound discs of area)

$$b : \text{Ham}_{\underline{L}}(\mathbb{D}) \twoheadrightarrow \mathcal{B}_k$$

Braid group on k strands: $(\gamma_1, \dots, \gamma_k) : [0,1] \to \mathbb{D}^{\times k}$

$i \neq j \Rightarrow \gamma_i(t) \neq \gamma_j(t) \quad \forall t \in [0,1]$

Setup II



$b(\varphi) = \sigma_1 \, \sigma_2^{-1}$

$\operatorname{lk} b(\varphi) = 0$

lk = "linking number".

A question

Can we bound from below the Hofer energy needed to realise a braid?

Equivalently, define pseudonorms: $\mathcal{B}_k \longrightarrow \mathbb{R}_{\geq 0}$,

$$g \longmapsto \|g\|_{\underline{L}} := \inf \left\{ \|\varphi\|_{Hofer} \;\middle|\; \begin{array}{l} \varphi \in Ham_{\underline{L}}(\mathbb{D}), \\ b(\varphi) = g \end{array} \right\}$$

Find a function $\mathcal{B}_k \to \mathbb{R}$ which is $\|\cdot\|_{\underline{L}}$ - Lipschitz.

Main result

Assume each of the k circles bounds a disc of area $A \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$

Then: $\quad \forall \varphi \in Ham_{\underline{L}}(D), \quad \|\varphi\| \geq C(k,A) \cdot |\mathcal{L}_k\, b(\varphi)|$

$\& \quad \|g\|_{\underline{L}} \geq C(k,A) \cdot |\mathcal{L}_k\, b(g)|$

Some remarks

$$\|\varphi\| \geq C(k,A)\cdot|\ell k\, b(\varphi)| \quad \& \quad \|g\|_L \geq C(k,A)\cdot|\ell k\, b(g)|$$

- $C(k,A)$ is explicit.
- New persistence phenomenon: if $\phi, \psi \in Ham_L(\mathbb{D})$,

$$d_H(\phi,\psi) \geq C(k,A)\cdot|\ell k\, b(\varphi) - \ell k\, b(\psi)|$$

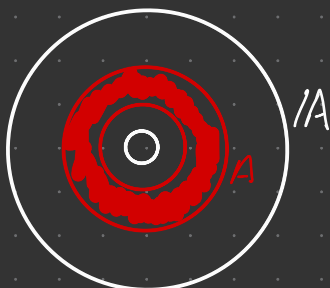
- If $k=2$, we already have non-degeneracy for $\|\cdot\|_L$. $k\geq 3$, it was proved by Chen '23.

  (if $b(\phi)\neq b(\psi)$, $d_H(\phi,\psi)\geq \varepsilon(L)$).

- Diffeomorphisms of zero-Calabi with arbitrary (asymptotic) Hofer norm.

Earlier results

Le Roux ('10)

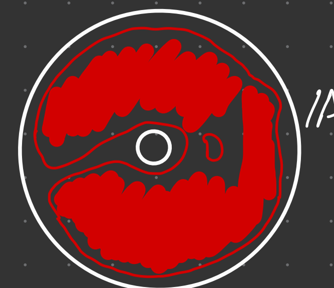


$\varphi \in Ham_c(\mathbb{A})$, $\varphi(A) = A$

notion of rotation number $r(\varphi, A)$

$\Rightarrow \|\varphi\| \geq C \cdot |r(\varphi, A)|$

Khanevsky ('11)



$\varphi \in Ham_c(\mathbb{A})$, $\varphi(D) = D$, $\mathrm{Area}(D) > 1/2$

notion of rotation number $r(\varphi, D)$

$\Rightarrow \|\varphi\| \geq C \cdot |r(\varphi, D)|$

Tools

Quantitative Heegaard-Floer Homology :

Σ symplectic surface $\Rightarrow$ $\mathfrak{S}_k \circlearrowright \Sigma^{\times k}$ by permutation of coordinates

$\Rightarrow$ $\mathrm{Sym}^k \Sigma$ Kähler, and

$\underline{L} = L_1, \dots, L_k \subset \Sigma$ disjoint embedded circles + (Area conditions)

$\Rightarrow$ $\mathrm{Sym}^k \underline{L} \subseteq \mathrm{Sym}^k \Sigma$ monotone lagrangian torus.

$\phi_H^1 =: \varphi \in \mathrm{Ham}(\Sigma) \rightsquigarrow \mathrm{Sym}^k \varphi \in \mathrm{Ham}(\mathrm{Sym}^k \Sigma)$ generated by $\mathrm{Sym}^k H_t([x_1, \dots, x_k]) := \sum_{i=1}^k H_t(x_i)$

$\Rightarrow$ $CF(\underline{L}, H) := CF(\mathrm{Sym}^k \underline{L}, \mathrm{Sym}^k H)$

Idea of Proof

$CF(\underline{L}, H) \Rightarrow$ Hofer-Lipschitz spectral invariants $c_{\underline{L}}(\varphi)$.

Generators: capped intersection points $\mathrm{Sym}^k \underline{L} \cap (\mathrm{Sym}^k \varphi)\, \mathrm{Sym}^k \underline{L}$

Capping: homotopy trivial braid $\to b(\varphi)^{-1}$.

Action: $\mathcal{A}^{\eta}_{H}(\hat{y}) = cst - \int_{\mathbb{D}} \hat{y}^* \omega - \eta\, [\hat{y}] \cdot \Delta$ $\qquad \Delta \subset \mathrm{Sym}^k \Sigma$ diagonal.

η = monotonicity parameter

$[\hat{y}] \cdot \Delta$ to be related to $\mathrm{lk}\, b(\varphi)$ and then extracted from spectral invariants:

embed $\mathbb{D} \hookrightarrow S^2(1+s_i)$ $\quad i = 1, 2$

$\underline{L} \rightsquigarrow \underline{L}_i \quad \Rightarrow \eta_1 \neq \eta_2$

$H \rightsquigarrow H_i$

$$\left.\right\} c_{\underline{L}_1}(\varphi_1) - c_{\underline{L}_2}(\varphi_2) = (\eta_2 - \eta_1)[\hat{y}] \cdot \Delta$$

Idea of Proof II

Biholomorphism $\mathrm{Sym}^k \mathbb{S}^2(1+s_1) \xrightarrow{\cong} \mathrm{Sym}^k \mathbb{S}^2(1+s_2)$

$$CF(\underline{L}_1, H_1) \cong CF(\underline{L}_2, H_2)$$

<u>not</u> as filtered complexes.

Uniform shift in filtration $= (\eta_2 - \eta_1)[\hat{y}] \cdot \Delta =$ RHS of the equality to prove.

$$\| \\ c_{\underline{L}_1}(\varphi_1) - c_{\underline{L}_2}(\varphi_2)$$

If we change cappings properly:

$$c_{\underline{L}_1}(\varphi_1) - c_{\underline{L}_2}(\varphi_2) = \cancel{cst} - \cancel{cst} + \cancel{\int_{D} \hat{y}^* \omega} - \cancel{\int_{D} \hat{y}^* \omega} + (\eta_2 - \eta_1)[\hat{y}] \cdot \Delta$$

Thank you, and

Happy New Year!