Riemannian distance and symplectic embeddings to cotangent bundle

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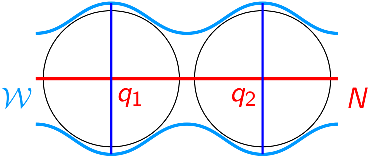
Motivated by the (relative) ball packing:

Definition

N closed, $\mathcal{W}$ (bounded) neighbourhood of $N \subset T^*N$. $q_1, q_2 \in N$

$$\rho_{\mathcal{W}}(q_1, q_2) := \sup \left\{ \frac{\pi r^2}{2} \;\middle|\; \begin{array}{l} \exists e : B_1^{2n}(r) \sqcup B_2^{2n}(r) \to \mathcal{W}, \quad e^*\omega_{can} = \omega_{st}, \\ e^{-1}(N) = B_1^n(r) \times \{0\} \sqcup B_2^n(r) \times \{0\}, \\ e^{-1}\left(\bigsqcup T_{q_i}^* N \right) = \bigsqcup \{0\} \times B_i^n(r). \end{array} \right\}.$$



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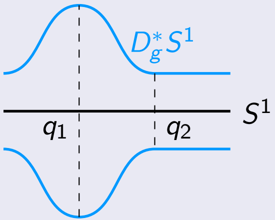

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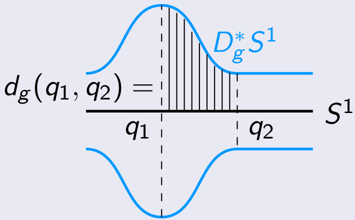

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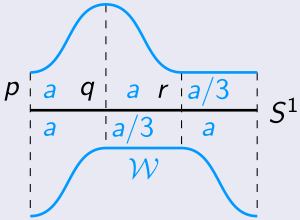

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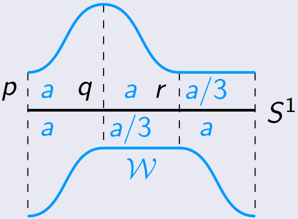

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$\gamma : [0,1] \to N$ piecewise smooth curve

Definition

$$L_{\rho_{\mathcal{W}}}(\gamma) = \sup_{\mathcal{P}} \sum_{1 \leq i \leq k} \rho_{\mathcal{W}}(\gamma(t_i), \gamma(t_{i+1})).$$

Set $\mathcal{W} = D^*_g N$, where $D^*_g N := \{p \in T^*N \mid g^*(p,p) < 1\}$, Set $\mathcal{W} = D_g^*N$, where $D_g^*N := \{p \in T^*N \mid g^*(p,p) < 1\}$,

Theorem (B. '23)

$$L_{\rho_{D_g^*N}}(\gamma) = \int_0^1 \|\gamma'(t)\|_g dt.$$

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Remark

Even if the boundary of the neighborhood $\mathcal{W}$ is smooth, the associated length structure doesn't need to be smooth.

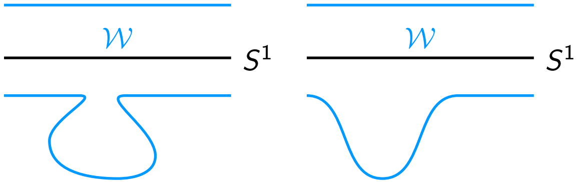

Figure: On the left: $L_{\rho_{\mathcal{W}}}$ is not C^0; on the right: $L_{\rho_{\mathcal{W}}}$ is not differentiable.

Proposition (B. '23)

If $e : B^{2n}(r_1) \sqcup B^{2n}(r_2) \to D^*_g N$ is a symplectic embedding relative to N such that

$$e^{-1}(D^*_{q_1} N \sqcup D^*_{q_2} N) = \{0\} \times B^n(r_1) \sqcup \{0\} \times B^n(r_2)$$

then

$$\pi r_1^2 + \pi r_2^2 \leq 4 d_g(q_1, q_2)$$

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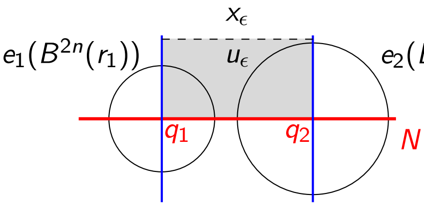

$$\begin{cases} u_\epsilon : [0, +\infty) \times [0, 1] \to T^* N, \\ \partial_s u_\epsilon + J_t(\partial_t u_\epsilon - X_{H_\epsilon}) = 0, \\ u_\epsilon(0, t) \in N, \\ u_\epsilon(s, i) \in T^*_{q_{i+1}} N. \end{cases}$$

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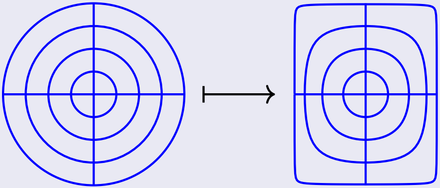
WLOG $a = b = 1$, set $f(q) := \int_0^q \sqrt{4/\pi - t^2}\,dt$



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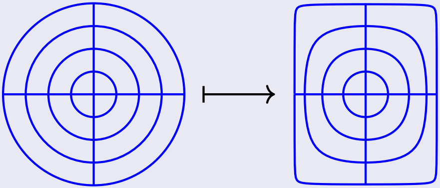

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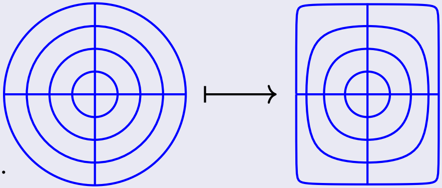

Definition

Normalized symplectic capacity is $c : \mathcal{O}p(\mathbb{R}^{2n}) \to [0, +\infty]$ s.t.

- (*Conformality*) $c(aX) = a^2 c(X)$,
- (*Monotonicity*) If there is a symplectic embedding $\psi : X_1 \to X_2$ then $c(X_1) \leq c(X_2)$,
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$$Gr(X) = \sup\{\pi r^2 \mid e : B^{2n}(r) \to X, e^*\omega_{st} = \omega_{st}\}$$

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c any normalized capacity, $\forall X \subset \mathbb{R}^{2n}$

$$Gr(X) \leq c(X) \leq c_Z(X)$$

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$$\Pi_1 : P_L(a, b) \to \mathbb{R}^2, \ \ \Pi_1(\mathbf{q}, \mathbf{p}) := (q_1, p_1)$$

$$\Pi_1(P_L(a, b)) = [-a, a] \times [-b, b] \Rightarrow c_Z(P_L(a, b)) \leq 4ab.$$

□

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$$\tilde{G}r(N; D^*_g N) := \sup \left\{ \pi r^2 \;\middle|\; \begin{array}{l} \exists e : B^{2n}(r) \to D^*_g N, \quad e^*\omega = \omega_{st}, \\ e^{-1}(N) = B^n(r) \times \{0\}, \\ e^{-1}(T^*_{e(0)} N) = \{0\} \times B^n(r). \end{array} \right\}$$

Recall: $\pi r_1^2 + \pi r_2^2 \leq 4d_g(q_1, q_2)$

Corollary

$$\tilde{G}r(N, D_g^*N) \leq 4diam_g(N).$$

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$\mathcal{W} \subset T^* N$ star-shaped domain
$\mathfrak{B} : N \times S^1 \to N$, non-contractible S^1 action

$$\kappa(\mathfrak{B}, \mathcal{W}) := \sup \left\{ \int x^* \lambda \mid x(t) \in \mathcal{W}, \text{ lift of } \mathfrak{B}(p, t) \right\}$$

Theorem (B.-Cant '23)

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Result is sharp for $N := T^n = \mathbb{R}/a_1\mathbb{Z} \times \cdots \times \mathbb{R}/a_n\mathbb{Z}$ and $\mathcal{W} = D_g^* T^n$.

Finsler case

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K is a unit ball in $\mathbb{R}^n$, and K^0 is it's polar body (unit ball with respect to the dual norm.)
For the lower bound we need an embedding:

$$\begin{aligned} & e : B^{2n}(2/\sqrt{\pi}) \to K \times K^0 \\ \text{s.t.} \quad & e^{-1}(K \times \{0\}) = B^n(2/\sqrt{\pi}) \times \{0\}, \\ & e^{-1}(\{0\} \times K^0) = \{0\} \times B^n(2/\sqrt{\pi}). \end{aligned}$$

Such an embedding would resolve strong Viterbo conjecture for $K \times K^0$, and hence Mahler's conjecture in all dimensions.

Thank You!